6

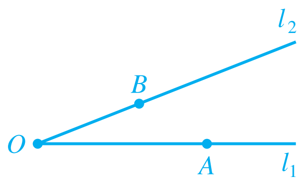
The Trigonometric Functions

Trigonometry was invented over 2000 years ago by the Greeks, who needed precise methods for measuring angles and sides of triangles. In fact, the word *trigonometry* was derived from the two Greek words *trigonon* (triangle) and *metria* (measurement). This chapter begins with a discussion of angles and how they are measured. We next introduce the trigonometric functions by using ratios of sides of a right triangle. After extending the domains of the trigonometric functions to arbitrary angles and real numbers, we consider their graphs and graphing techniques that make use of amplitudes, periods, and phase shifts. The chapter concludes with a section on applied problems.

6.1 Angles

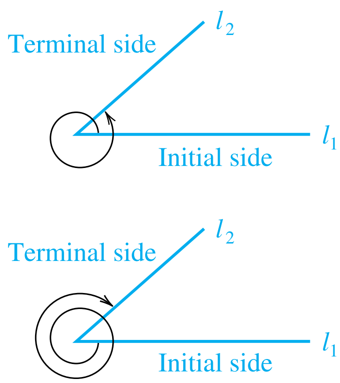
In geometry an **angle** is defined as the set of points determined by two rays, or half-lines, l_1 and l_2, having the same endpoint O. If A and B are points on l_1 and l_2, as in Figure 1, we refer to **angle *AOB*** (denoted $\angle AOB$). An angle may also be considered as two finite line segments with a common endpoint.

Figure 1



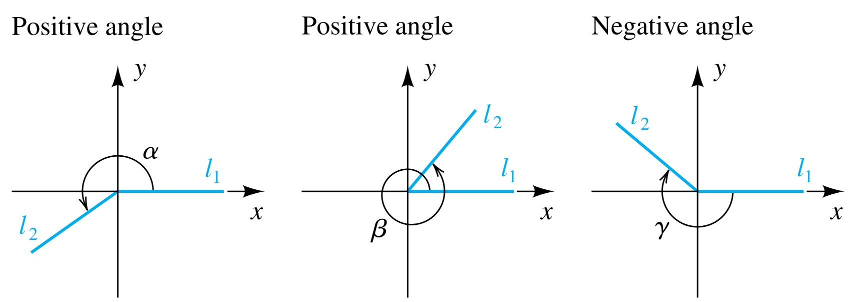
In trigonometry we often interpret angles as rotations of rays. Start with a fixed ray l_1, having endpoint O, and rotate it about O, in a plane, to a position specified by ray l_2. We call l_1 the **initial side,** l_2 the **terminal side,** and O the **vertex** of $\angle AOB$. The amount or direction of rotation is not restricted in any way. We might let l_1 make several revolutions in either direction about O before coming to position l_2, as illustrated by the curved arrows in Figure 2. Thus, many different angles have the same initial and terminal sides. Any two such angles are called **coterminal angles.** A **straight angle** is an angle whose sides lie on the same straight line but extend in opposite directions from its vertex.

Figure 2
Coterminal angles



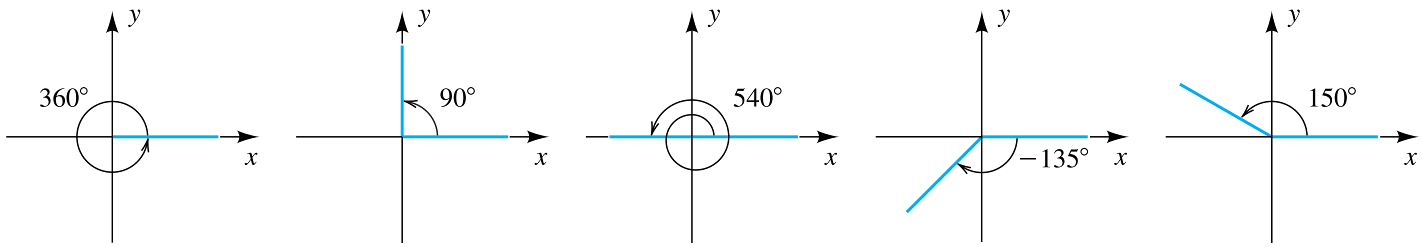
If we introduce a rectangular coordinate system, then the **standard position** of an angle is obtained by taking the vertex at the origin and letting the initial side l_1 coincide with the positive x-axis. If l_1 is rotated in a *counterclockwise* direction to the terminal position l_2, then the angle is considered **positive.** If l_1 is rotated in a *clockwise* direction, the angle is **negative.** We often denote angles by lowercase Greek letters such as α (*alpha*), β (*beta*), γ (*gamma*), θ (*theta*), ϕ (*phi*), and so on. Figure 3 contains sketches of two positive angles, α and β, and a negative angle, γ. If the terminal side of an angle in standard position is in a certain quadrant, we say that the *angle* is in that quadrant. In Figure 3, α is in quadrant III, β is in quadrant I, and γ is in quadrant II. An angle is called a **quadrantal angle** if its terminal side lies on a coordinate axis.

Figure 3 Standard position of an angle



One unit of measurement for angles is the **degree.** The angle in standard position obtained by one complete revolution in the counterclockwise direction has measure 360 degrees, written 360°. Thus, an angle of measure 1 degree (1°) is obtained by $\frac{1}{360}$ of one complete counterclockwise revolution. In Figure 4, several angles measured in degrees are shown in standard position on rectangular coordinate systems. Note that the first three are quadrantal angles.

Figure 4



Throughout our work, a notation such as $\theta = 60°$ specifies an angle θ whose measure is 60°. We also refer to *an angle of* 60° or *a* 60° *angle,* instead of using the more precise (but cumbersome) phrase *an angle having measure* 60°.

EXAMPLE 1 **Finding coterminal angles**

If $\theta = 60°$ is in standard position, find two positive angles and two negative angles that are coterminal with θ.

SOLUTION The angle θ is shown in standard position in the first sketch in Figure 5. To find positive coterminal angles, we may add 360° or 720° (or any other positive integer multiple of 360°) to θ, obtaining

$$60° + 360° = 420° \qquad \text{and} \qquad 60° + 720° = 780°.$$

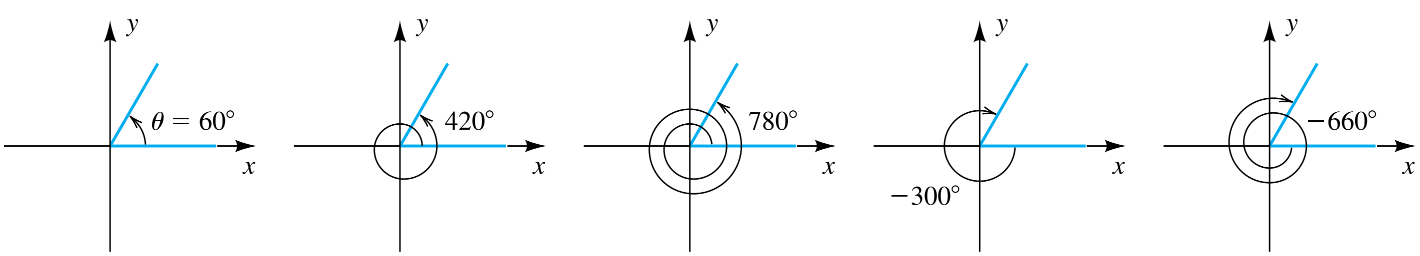
These coterminal angles are also shown in Figure 5.

To find negative coterminal angles, we may add −360° or −720° (or any other negative integer multiple of 360°), obtaining

$$60° + (-360°) = -300° \qquad \text{and} \qquad 60° + (-720°) = -660°,$$

as shown in the last two sketches in Figure 5.

Figure 5



A **right angle** is half of a straight angle and has measure 90°. The following chart contains definitions of other special types of angles.

Terminology	Definition	Illustrations
acute angle $\boldsymbol{\theta}$	$0° < \theta < 90°$	12°; 37°
obtuse angle $\boldsymbol{\theta}$	$90° < \theta < 180°$	95°; 157°
complementary angles $\boldsymbol{\alpha}$**,** $\boldsymbol{\beta}$	$\alpha + \beta = 90°$	20°, 70°; 7°, 83°
supplementary angles $\boldsymbol{\alpha}$**,** $\boldsymbol{\beta}$	$\alpha + \beta = 180°$	115°, 65°; 18°, 162°

If smaller measurements than the degree are required, we can use tenths, hundredths, or thousandths of degrees. Alternatively, we can divide the degree into 60 equal parts, called **minutes** (denoted by $'$), and each minute into 60 equal parts, called **seconds** (denoted by $''$). Thus, $1° = 60'$, and $1' = 60''$. The notation $\theta = 73°56'18''$ refers to an angle θ that has measure 73 degrees, 56 minutes, 18 seconds.

EXAMPLE 2 Finding complementary angles

Find the angle that is complementary to θ:

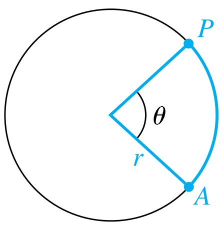
(a) $\theta = 25°43'37''$ **(b)** $\theta = 73.26°$

SOLUTION We wish to find $90° - \theta$. It is convenient to write 90° as an equivalent measure, $89°59'60''$.

(a)
$$\begin{aligned} 90° &= 89°59'60'' \\ \underline{\quad\theta\quad} &= \underline{25°43'37''} \\ 90° - \theta &= 64°16'23'' \end{aligned}$$

(b)
$$\begin{aligned} 90° &= 90.00° \\ \underline{\quad\theta\quad} &= \underline{73.26°} \\ 90° - \theta &= 16.74° \end{aligned}$$

Figure 6
Central angle θ



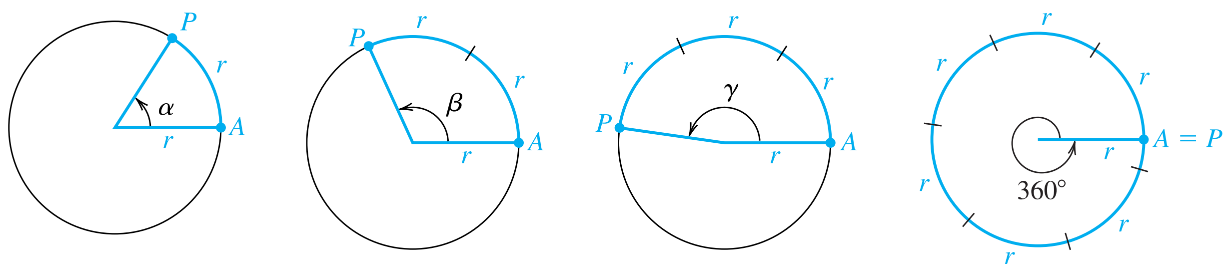
Degree measure for angles is used in applied areas such as surveying, navigation, and the design of mechanical equipment. In scientific applications that require calculus, it is customary to employ *radian measure.* To define an angle of radian measure 1, we consider a circle of any radius r. A **central angle** of a circle is an angle whose vertex is at the center of the circle. If θ is the central angle shown in Figure 6, we say that the **arc** $\boldsymbol{AP}$ (denoted $\overset{\frown}{AP}$) of the circle **subtends** θ or that θ **is subtended by** $\overset{\frown}{AP}$. If the length of $\overset{\frown}{AP}$ is equal to the radius r of the circle, then θ has a measure of one radian, as in the next definition.

Definition of Radian Measure	**One radian** is the measure of the central angle of a circle subtended by an arc equal in length to the radius of the circle.

If we consider a circle of radius r, then an angle α whose measure is 1 radian intercepts an arc AP of length r, as illustrated in Figure 7(a). The angle β in Figure 7(b) has radian measure 2, since it is subtended by an arc of length $2r$. Similarly, γ in (c) of the figure has radian measure 3, since it is subtended by an arc of length $3r$.

Figure 7

(a) $\alpha = 1$ radian **(b)** $\beta = 2$ radians **(c)** $\gamma = 3$ radians **(d)** $360° = 2\pi \approx 6.28$ radians



To find the radian measure corresponding to 360°, we must find the number of times that a circular arc of length r can be laid off along the circumference (see Figure 7(d)). This number is not an integer or even a rational number. Since the circumference of the circle is $2\pi r$, the number of times r units can be laid off is 2π. Thus, an angle of measure 2π radians corresponds to the degree measure 360°, and we write $360° = 2\pi$ radians. This result gives us the following relationships.

Relationships Between Degrees and Radians

(1) $180° = \pi$ radians

(2) $1° = \dfrac{\pi}{180}$ radian ≈ 0.0175 radian

(3) 1 radian $= \left(\dfrac{180°}{\pi}\right) \approx 57.2958°$

When radian measure of an angle is used, no units will be indicated. Thus, if an angle has radian measure 5, we write $\theta = 5$ instead of $\theta = 5$ *radians.* There should be no confusion as to whether radian or degree measure is being used, since if θ has *degree* measure 5°, we write $\theta = 5°$, and *not* $\theta = 5$.

The next chart illustrates how to change from one angular measure to another.

Changing Angular Measures

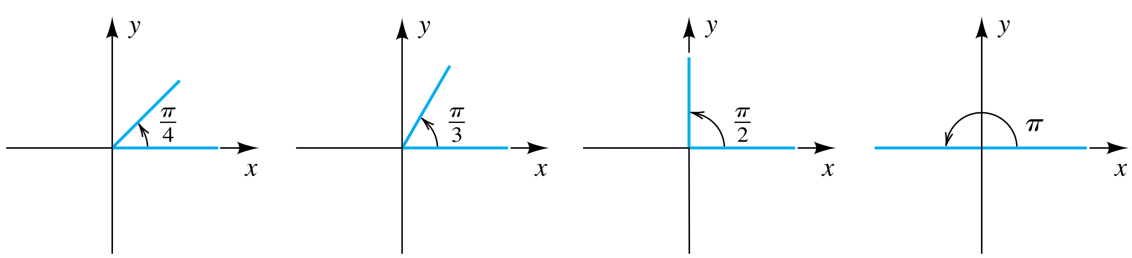
To change	Multiply by	Illustrations
degrees to radians	$\dfrac{\pi}{180^\circ}$	$150^\circ = 150^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{5\pi}{6}$ $225^\circ = 225^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{5\pi}{4}$
radians to degrees	$\dfrac{180^\circ}{\pi}$	$\dfrac{7\pi}{4} = \dfrac{7\pi}{4}\left(\dfrac{180^\circ}{\pi}\right) = 315^\circ$ $\dfrac{\pi}{3} = \dfrac{\pi}{3}\left(\dfrac{180^\circ}{\pi}\right) = 60^\circ$

We may use the techniques illustrated in the preceding chart to obtain the following table, which displays the corresponding radian and degree measures of special angles.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°

Several of these special angles, in radian measure, are shown in standard position in Figure 8.

Figure 8



EXAMPLE 3 Changing radians to degrees, minutes, and seconds

If $\theta = 3$, approximate θ in terms of degrees, minutes, and seconds.

SOLUTION

$$
\begin{aligned}
3 \text{ radians} &= 3\left(\frac{180^\circ}{\pi}\right) && \text{multiply by } \frac{180^\circ}{\pi} \\
&\approx 171.8873^\circ && \text{approximate} \\
&= 171^\circ + (0.8873)(60') && 1^\circ = 60' \\
&= 171^\circ + 53.238' && \text{multiply} \\
&= 171^\circ + 53' + (0.238)(60'') && 1' = 60'' \\
&= 171^\circ 53' + 14.28'' && \text{multiply} \\
&\approx 171^\circ 53' 14'' && \text{approximate}
\end{aligned}
$$

EXAMPLE 4 **Expressing minutes and seconds as decimal degrees**

Express $19^\circ 47' 23''$ as a decimal, to the nearest ten-thousandth of a degree.

SOLUTION Since $1' = \left(\frac{1}{60}\right)^\circ$ and $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ$,

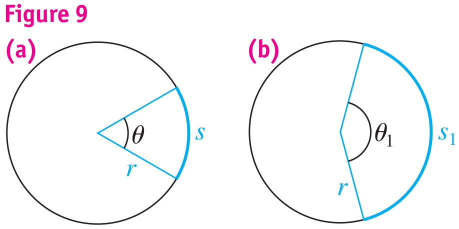
$$
\begin{aligned}
19^\circ 47' 23'' &= 19^\circ + \left(\tfrac{47}{60}\right)^\circ + \left(\tfrac{23}{3600}\right)^\circ \\
&\approx 19^\circ + 0.7833^\circ + 0.0064^\circ \\
&= 19.7897^\circ.
\end{aligned}
$$

The next result specifies the relationship between the length of a circular arc and the central angle that it subtends.

Formula for the Length of a Circular Arc	If an arc of length s on a circle of radius r subtends a central angle of radian measure θ, then $$s = r\theta.$$

A mnemonic device for remembering $s = r\theta$ is SRO (Standing Room Only).

Figure 9



PROOF A typical arc of length s and the corresponding central angle θ are shown in Figure 9(a). Figure 9(b) shows an arc of length s_1 and central angle θ_1. If radian measure is used, then, from plane geometry, the ratio of the lengths of the arcs is the same as the ratio of the angular measures; that is,

$$\frac{s}{s_1} = \frac{\theta}{\theta_1}, \qquad \text{or} \qquad s = \frac{\theta}{\theta_1} s_1.$$

(continued)

If we consider the special case in which θ_1 has radian measure 1, then, from the definition of radian, $s_1 = r$ and the last equation becomes

$$s = \frac{\theta}{1} \cdot r = r\theta.$$

Notice that if $\theta = 2\pi$, then the formula for the length of a circular arc becomes $s = r(2\pi)$, which is simply the formula for the circumference of a circle, $C = 2\pi r$.

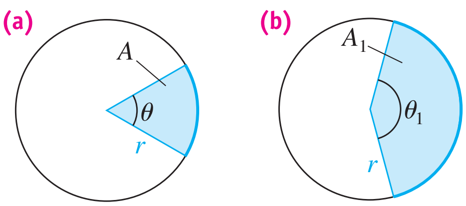
The next formula is proved in a similar manner.

Formula for the Area of a Circular Sector

If θ is the radian measure of a central angle of a circle of radius r and if A is the area of the circular sector determined by θ, then

$$A = \tfrac{1}{2}r^2\theta.$$

Figure 10



PROOF If A and A_1 are the areas of the sectors in Figures 10(a) and 10(b), respectively, then, from plane geometry,

$$\frac{A}{A_1} = \frac{\theta}{\theta_1}, \qquad \text{or} \qquad A = \frac{\theta}{\theta_1}A_1.$$

If we consider the special case $\theta_1 = 2\pi$, then $A_1 = \pi r^2$ and

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta.$$

When using the preceding formulas, it is important to remember to use the radian measure of θ rather than the degree measure, as illustrated in the next example.

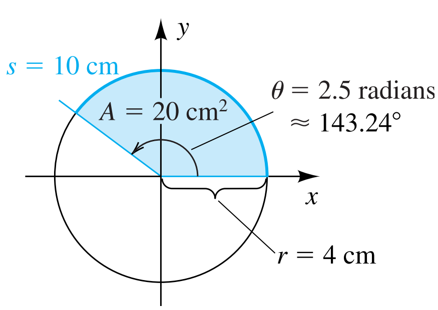
EXAMPLE 5 **Using the circular arc and sector formulas**

In Figure 11, a central angle θ is subtended by an arc 10 centimeters long on a circle of radius 4 centimeters.

(a) Approximate the measure of θ in degrees.

(b) Find the area of the circular sector determined by θ.

Figure 11



SOLUTION We proceed as follows:

(a) $s = r\theta$ length of a circular arc formula

$\theta = \dfrac{s}{r}$ solve for θ

$= \frac{10}{4} = 2.5$ let $s = 10$, $r = 4$

This is the *radian* measure of θ. Changing to degrees, we have

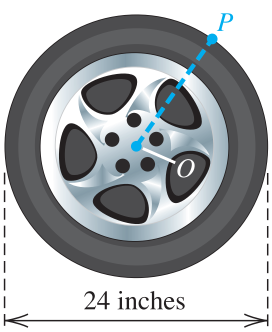
$$\theta = 2.5\left(\frac{180^\circ}{\pi}\right) = \frac{450^\circ}{\pi} \approx 143.24^\circ.$$

(b) $A = \frac{1}{2}r^2\theta$ — area of a circular sector formula

$= \frac{1}{2}(4)^2(2.5)$ — let $r = 4$, $\theta = 2.5$ radians

$= 20\text{ cm}^2$ — multiply

Figure 12



The **angular speed** of a wheel that is rotating at a constant rate is the angle generated in one unit of time by a line segment from the center of the wheel to a point P on the circumference (see Figure 12). The **linear speed** of a point P on the circumference is the distance that P travels per unit of time. By dividing both sides of the formula for a circular arc by time t, we obtain a relationship for linear speed and angular speed; that is,

$$\frac{s}{t} = \frac{r\theta}{t}, \qquad \text{or, equivalently,} \qquad \underbrace{\frac{s}{t}}_{\text{linear speed}} = r \cdot \underbrace{\frac{\theta}{t}}_{\text{angular speed}}.$$

EXAMPLE 6 **Finding angular and linear speeds**

Suppose that the wheel in Figure 12 is rotating at a rate of 800 rpm (revolutions per minute).

(a) Find the angular speed of the wheel.

(b) Find the linear speed (in in./min and mi/hr) of a point P on the circumference of the wheel.

SOLUTION

(a) Let O denote the center of the wheel, and let P be a point on the circumference. Because the number of revolutions per minute is 800 and because each revolution generates an angle of 2π radians, the angle generated by the line segment OP in one minute has radian measure $(800)(2\pi)$; that is,

$$\text{angular speed} = \frac{800\text{ revolutions}}{1\text{ minute}} \cdot \frac{2\pi\text{ radians}}{1\text{ revolution}} = 1600\pi\text{ radians per minute.}$$

Note that the diameter of the wheel is irrelevant in finding the angular speed.

(b) linear speed $= \text{radius} \cdot \text{angular speed}$

$= (12\text{ in.})(1600\pi\text{ rad/min})$

$= 19{,}200\pi\text{ in./min}$

Converting in./min to mi/hr, we get

$$\frac{19{,}200\pi\text{ in.}}{1\text{ min}} \cdot \frac{60\text{ min}}{1\text{ hr}} \cdot \frac{1\text{ ft}}{12\text{ in.}} \cdot \frac{1\text{ mi}}{5280\text{ ft}} \approx 57.1\text{ mi/hr.}$$

Unlike the angular speed, the linear speed *is* dependent on the diameter of the wheel.

6.1 Exercises

Exer. 1–4: If the given angle is in standard position, find two positive coterminal angles and two negative coterminal angles.

1 (a) 120° (b) 135° (c) -30°

2 (a) 240° (b) 315° (c) -150°

3 (a) 620° (b) $\dfrac{5\pi}{6}$ (c) $-\dfrac{\pi}{4}$

4 (a) 570° (b) $\dfrac{2\pi}{3}$ (c) $-\dfrac{5\pi}{4}$

Exer. 5–6: Find the angle that is complementary to θ.

5 (a) $\theta = 5^\circ17'34''$ (b) $\theta = 32.5^\circ$

6 (a) $\theta = 63^\circ4'15''$ (b) $\theta = 82.73^\circ$

Exer. 7–8: Find the angle that is supplementary to θ.

7 (a) $\theta = 48^\circ51'37''$ (b) $\theta = 136.42^\circ$

8 (a) $\theta = 152^\circ12'4''$ (b) $\theta = 15.9^\circ$

Exer. 9–12: Find the exact radian measure of the angle.

9 (a) 150° (b) -60° (c) 225°

10 (a) 120° (b) -135° (c) 210°

11 (a) 450° (b) 72° (c) 100°

12 (a) 630° (b) 54° (c) 95°

Exer. 13–16: Find the exact degree measure of the angle.

13 (a) $\dfrac{2\pi}{3}$ (b) $\dfrac{11\pi}{6}$ (c) $\dfrac{3\pi}{4}$

14 (a) $\dfrac{5\pi}{6}$ (b) $\dfrac{4\pi}{3}$ (c) $\dfrac{11\pi}{4}$

15 (a) $-\dfrac{7\pi}{2}$ (b) 7π (c) $\dfrac{\pi}{9}$

16 (a) $-\dfrac{5\pi}{2}$ (b) 9π (c) $\dfrac{\pi}{16}$

Exer. 17–20: Express θ in terms of degrees, minutes, and seconds, to the nearest second.

17 $\theta = 2$

18 $\theta = 1.5$

19 $\theta = 5$

20 $\theta = 4$

Exer. 21–24: Express the angle as a decimal, to the nearest ten-thousandth of a degree.

21 $37^\circ41'$

22 $83^\circ17'$

23 $115^\circ26'27''$

24 $258^\circ39'52''$

Exer. 25–28: Express the angle in terms of degrees, minutes, and seconds, to the nearest second.

25 63.169°

26 12.864°

27 310.6215°

28 81.7238°

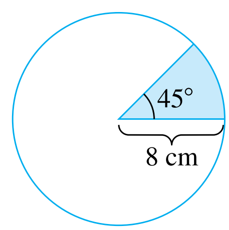
Exer. 29–30: If a circular arc of the given length s subtends the central angle θ on a circle, find the radius of the circle.

29 $s = 10$ cm, $\theta = 4$

30 $s = 3$ km, $\theta = 20^\circ$

Exer. 31–32: (a) Find the length of the arc of the colored sector in the figure. (b) Find the area of the sector.

31



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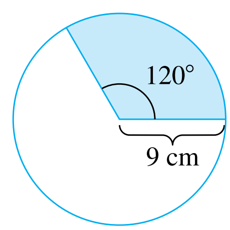


Exer. 33–34: (a) Find the radian and degree measures of the central angle θ subtended by the given arc of length s on a circle of radius r. (b) Find the area of the sector determined by θ.

33 $s = 7$ cm, $r = 4$ cm

34 $s = 3$ ft, $r = 20$ in.

Exer. 35–36: (a) Find the length of the arc that subtends the given central angle θ on a circle of diameter d. (b) Find the area of the sector determined by θ.

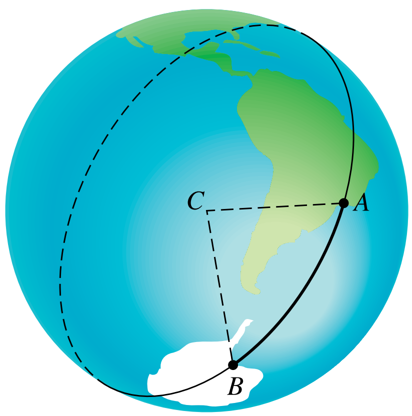
35 $\theta = 50^\circ$, $d = 16$ m

36 $\theta = 2.2$, $d = 120$ cm

37 Measuring distances on Earth The distance between two points A and B on Earth is measured along a circle having center C at the center of Earth and radius equal to the distance from C to the surface (see the figure). If the diameter of Earth is approximately 8000 miles, approximate the distance between A and B if angle ACB has the indicated measure:

(a) $60°$ (b) $45°$ (c) $30°$ (d) $10°$ (e) $1°$

Exercise 37



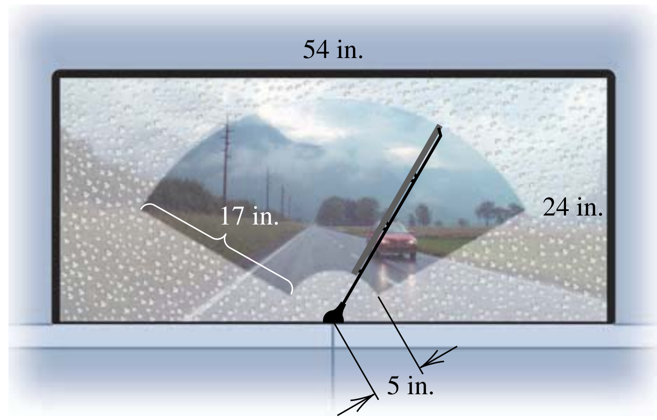
38 Nautical miles Refer to Exercise 37. If angle ACB has measure $1'$, then the distance between A and B is a nautical mile. Approximate the number of land (statute) miles in a nautical mile.

39 Measuring angles using distance Refer to Exercise 37. If two points A and B are 500 miles apart, express angle ACB in radians and in degrees.

40 A hexagon is inscribed in a circle. If the difference between the area of the circle and the area of the hexagon is 24 m^2, use the formula for the area of a sector to approximate the radius r of the circle.

41 Window area A rectangular window measures 54 inches by 24 inches. There is a 17-inch wiper blade attached by a 5-inch arm at the center of the base of the window, as shown in the figure. If the arm rotates 120°, approximate the percentage of the window's area that is wiped by the blade.

Exercise 41



42 A tornado's core A simple model of the core of a tornado is a right circular cylinder that rotates about its axis. If a tornado has a core diameter of 200 feet and maximum wind speed of 180 mi/hr (or 264 ft/sec) at the perimeter of the core, approximate the number of revolutions the core makes each minute.

43 Earth's rotation Earth rotates about its axis once every 23 hours, 56 minutes, and 4 seconds. Approximate the number of radians Earth rotates in one second.

44 Earth's rotation Refer to Exercise 43. The equatorial radius of Earth is approximately 3963.3 miles. Find the linear speed of a point on the equator as a result of Earth's rotation.

Exer. 45–46: A wheel of the given radius is rotating at the indicated rate.

(a) Find the angular speed (in radians per minute).

(b) Find the linear speed of a point on the circumference (in ft/min).

45 radius 5 in., 40 rpm

46 radius 9 in., 2400 rpm

47 Rotation of compact discs (CDs) The drive motor of a particular CD player is controlled to rotate at a speed of 200 rpm when reading a track 5.7 centimeters from the center of the CD. The speed of the drive motor must vary so that the reading of the data occurs at a constant rate.

(a) Find the angular speed (in radians per minute) of the drive motor when it is reading a track 5.7 centimeters from the center of the CD.

(b) Find the linear speed (in cm/sec) of a point on the CD that is 5.7 centimeters from the center of the CD.

(c) Find the angular speed (in rpm) of the drive motor when it is reading a track 3 centimeters from the center of the CD.

(d) Find a function S that gives the drive motor speed in rpm for any radius r in centimeters, where $2.3 \le r \le 5.9$. What type of variation exists between the drive motor speed and the radius of the track being read? Check your answer by graphing S and finding the speeds for $r = 3$ and $r = 5.7$.

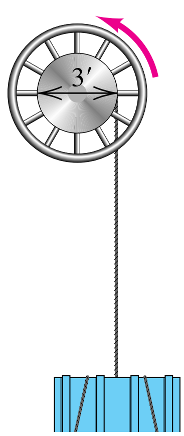
48 Tire revolutions A typical tire for a compact car is 22 inches in diameter. If the car is traveling at a speed of 60 mi/hr, find the number of revolutions the tire makes per minute.

49 Cargo winch A large winch of diameter 3 feet is used to hoist cargo, as shown in the figure.

(a) Find the distance the cargo is lifted if the winch rotates through an angle of radian measure $7\pi/4$.

(b) Find the angle (in radians) through which the winch must rotate in order to lift the cargo d feet.

Exercise 49

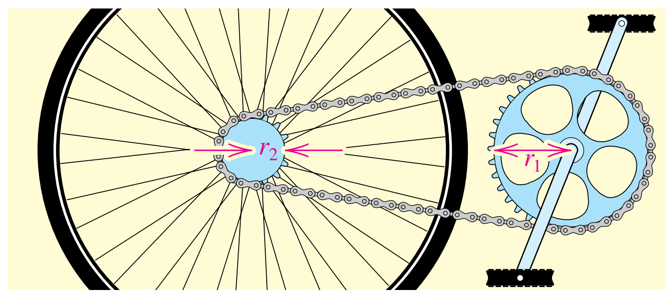


50 Pendulum's swing A pendulum in a grandfather clock is 4 feet long and swings back and forth along a 6-inch arc. Approximate the angle (in degrees) through which the pendulum passes during one swing.

51 Pizza values A vender sells two sizes of pizza by the slice. The *small* slice is $\frac{1}{6}$ of a circular 18-inch-diameter pizza, and it sells for \$2.00. The *large* slice is $\frac{1}{8}$ of a circular 26-inch-diameter pizza, and it sells for \$3.00. Which slice provides more pizza per dollar?

52 Bicycle mechanics The sprocket assembly for a bicycle is shown in the figure. If the sprocket of radius r_1 rotates through an angle of θ_1 radians, find the corresponding angle of rotation for the sprocket of radius r_2.

Exercise 52



53 Bicycle mechanics Refer to Exercise 52. An expert cyclist can attain a speed of 40 mi/hr. If the sprocket assembly has $r_1 = 5$ in., $r_2 = 2$ in., and the wheel has a diameter of 28 inches, approximately how many revolutions per minute of the front sprocket wheel will produce a speed of 40 mi/hr? (*Hint:* First change 40 mi/hr to in./sec.)

54 Magnetic pole drift The geographic and magnetic north poles have different locations. Currently, the magnetic north pole is drifting westward through 0.0017 radian per year, where the angle of drift has its vertex at the center of Earth. If this movement continues, approximately how many years will it take for the magnetic north pole to drift a total of 5°?

6.2 Trigonometric Functions of Angles

We shall introduce the trigonometric functions in the manner in which they originated historically—as ratios of sides of a right triangle. A triangle is a **right triangle** if one of its angles is a right angle. If θ is any acute angle, we may consider a right triangle having θ as one of its angles, as in Figure 1,

Figure 1

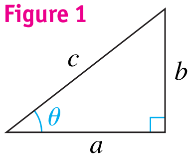
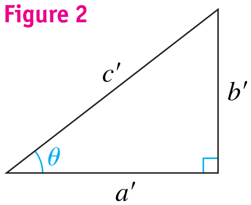


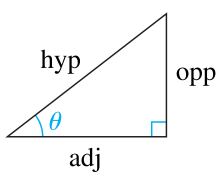
Figure 2



We will refer to these six trigonometric functions as **the trigonometric functions. Here are some other, less common trigonometric functions that we will not use in this text:*

$$\text{vers } \theta = 1 - \cos\theta$$
$$\text{covers } \theta = 1 - \sin\theta$$
$$\text{exsec } \theta = \sec\theta - 1$$
$$\text{hav } \theta = \tfrac{1}{2}\text{vers } \theta$$

Figure 3



where the symbol ⌐ specifies the 90° angle. Six ratios can be obtained using the lengths a, b, and c of the sides of the triangle:

$$\frac{b}{c}, \quad \frac{a}{c}, \quad \frac{b}{a}, \quad \frac{a}{b}, \quad \frac{c}{a}, \quad \frac{c}{b}$$

We can show that these ratios depend only on θ, and not on the size of the triangle, as indicated in Figure 2. Since the two triangles have equal angles, they are similar, and therefore ratios of corresponding sides are proportional. For example,

$$\frac{b}{c} = \frac{b'}{c'}, \quad \frac{a}{c} = \frac{a'}{c'}, \quad \frac{b}{a} = \frac{b'}{a'}.$$

Thus, for each θ, the six ratios are uniquely determined and hence are functions of θ. They are called the **trigonometric functions*** and are designated as the **sine, cosine, tangent, cotangent, secant,** and **cosecant** functions, abbreviated **sin, cos, tan, cot, sec,** and **csc,** respectively. The symbol $\sin(\theta)$, or $\sin\theta$, is used for the ratio b/c, which the sine function associates with θ. Values of the other five functions are denoted in similar fashion. To summarize, if θ is the acute angle of the right triangle in Figure 1, then, by definition,

$$\sin\theta = \frac{b}{c} \qquad \cos\theta = \frac{a}{c} \qquad \tan\theta = \frac{b}{a}$$
$$\csc\theta = \frac{c}{b} \qquad \sec\theta = \frac{c}{a} \qquad \cot\theta = \frac{a}{b}.$$

The domain of each of the six trigonometric functions is the set of all acute angles. Later in this section we will extend the domains to larger sets of angles, and in the next section, to real numbers.

If θ is the angle in Figure 1, we refer to the sides of the triangle of lengths a, b, and c as the **adjacent side, opposite side,** and **hypotenuse,** respectively. We shall use **adj, opp,** and **hyp** to denote the lengths of the sides. We may then represent the triangle as in Figure 3. With this notation, the trigonometric functions may be expressed as follows.

Definition of the Trigonometric Functions of an Acute Angle of a Right Triangle

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

A mnemonic device for remembering the top row in the definition is

SOH CAH TOA,

where SOH is an abbreviation for Sin θ = Opp/Hyp, *and so forth.*

The formulas in the preceding definition can be applied to any right triangle without attaching the labels a, b, c to the sides. Since the lengths of the sides of a triangle are positive real numbers, *the values of the six trigonometric functions are positive for every acute angle θ.* Moreover, the hypotenuse is always greater than the adjacent or opposite side, and hence $\sin\theta < 1$, $\cos\theta < 1$, $\csc\theta > 1$, and $\sec\theta > 1$ for every acute angle θ.

Note that since

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \quad \text{and} \quad \csc\theta = \frac{\text{hyp}}{\text{opp}},$$

$\sin\theta$ and $\csc\theta$ are reciprocals of each other, giving us the two identities in the left-hand column of the next box. Similarly, $\cos\theta$ and $\sec\theta$ are reciprocals of each other, as are $\tan\theta$ and $\cot\theta$.

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

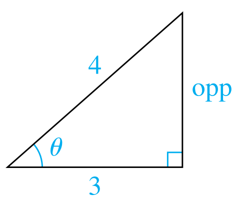
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Several other important identities involving the trigonometric functions will be discussed at the end of this section.

EXAMPLE 1 **Finding trigonometric function values**

If θ is an acute angle and $\cos\theta = \frac{3}{4}$, find the values of the trigonometric functions of θ.

Figure 4



SOLUTION We begin by sketching a right triangle having an acute angle θ with adj $= 3$ and hyp $= 4$, as shown in Figure 4, and proceed as follows:

$$\begin{aligned} 3^2 + (\text{opp})^2 &= 4^2 && \text{Pythagorean theorem} \\ (\text{opp})^2 &= 16 - 9 = 7 && \text{isolate } (\text{opp})^2 \\ \text{opp} &= \sqrt{7} && \text{take the square root} \end{aligned}$$

Applying the definition of the trigonometric functions of an acute angle of a right triangle, we obtain the following:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{7}}{4} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{4} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{7}}{3}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{4}{\sqrt{7}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{4}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{\sqrt{7}}$$

In Example 1 we could have rationalized the denominators for $\csc\theta$ and $\cot\theta$, writing

$$\csc\theta = \frac{4\sqrt{7}}{7} \quad \text{and} \quad \cot\theta = \frac{3\sqrt{7}}{7}.$$

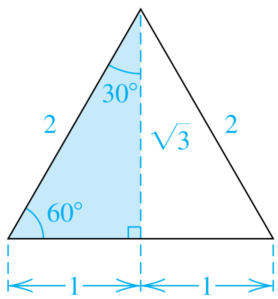
However, in most examples and exercises we will leave expressions in unrationalized form. An exception to this practice is the special trigonometric function values corresponding to 60°, 30°, and 45°, which are obtained in the following example.

EXAMPLE 2 **Finding trigonometric function values of 60°, 30°, and 45°**

Find the values of the trigonometric functions that correspond to θ:

(a) $\theta = 60°$ **(b)** $\theta = 30°$ **(c)** $\theta = 45°$

Figure 5



SOLUTION Consider an equilateral triangle with sides of length 2. The median from one vertex to the opposite side bisects the angle at that vertex, as illustrated by the dashes in Figure 5. By the Pythagorean theorem, the side opposite 60° in the shaded right triangle has length $\sqrt{3}$. Using the formulas for the trigonometric functions of an acute angle of a right triangle, we obtain the values corresponding to 60° and 30° as follows:

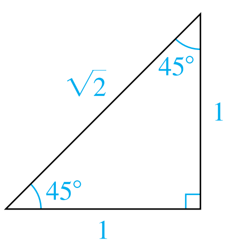
(a) $\sin 60° = \dfrac{\sqrt{3}}{2}$ $\quad \cos 60° = \dfrac{1}{2}$ $\quad \tan 60° = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

$\csc 60° = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ $\quad \sec 60° = \dfrac{2}{1} = 2$ $\quad \cot 60° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

(b) $\sin 30° = \dfrac{1}{2}$ $\quad \cos 30° = \dfrac{\sqrt{3}}{2}$ $\quad \tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\csc 30° = \dfrac{2}{1} = 2$ $\quad \sec 30° = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ $\quad \cot 30° = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

Figure 6



(c) To find the values for $\theta = 45°$, we may consider an isosceles right triangle whose two equal sides have length 1, as illustrated in Figure 6. By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{2}$. Hence, the values corresponding to 45° are as follows:

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos 45° \qquad \tan 45° = \frac{1}{1} = 1$$

$$\csc 45° = \frac{\sqrt{2}}{1} = \sqrt{2} = \sec 45° \qquad \cot 45° = \frac{1}{1} = 1$$

For reference, we list the values found in Example 2, together with the radian measures of the angles, in the following table. Two reasons for stressing these values are that they are exact and that they occur frequently in work involving trigonometry. Because of the importance of these special values, it is a good idea either to memorize the table or to learn to find the values quickly by using triangles, as in Example 2.

Special Values of the Trigonometric Functions

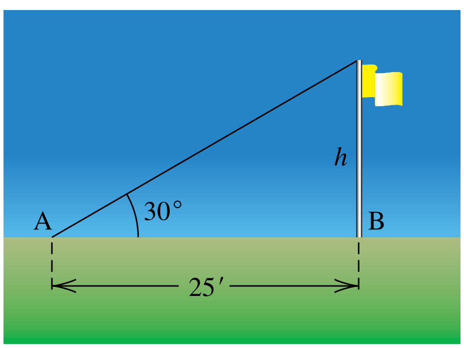
θ (radians)	θ (degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

The next example illustrates a practical use for trigonometric functions of acute angles. Additional applications involving right triangles will be considered in Section 6.7.

EXAMPLE 3 **Finding the height of a flagpole**

A surveyor observes that at a point A, located on level ground a distance 25.0 feet from the base B of a flagpole, the angle between the ground and the top of the pole is 30°. Approximate the height h of the pole to the nearest tenth of a foot.

Figure 7



SOLUTION Referring to Figure 7, we see that we want to relate the opposite side and the adjacent side, h and 25, respectively, to the 30° angle. This suggests that we use a trigonometric function involving those two sides—namely, tan or cot. It is usually easier to solve the problem if we select the function for which the variable is in the numerator. Hence, we have

$$\tan 30° = \frac{h}{25} \quad \text{or, equivalently,} \quad h = 25 \tan 30°.$$

We use the value of tan 30° from Example 2 to find h:

$$h = 25\left(\frac{\sqrt{3}}{3}\right) \approx 14.4 \text{ ft}$$

It is possible to approximate, to any degree of accuracy, the values of the trigonometric functions for any acute angle. Calculators have keys labeled SIN, COS, and TAN that can be used to approximate values of these functions. The values of csc, sec, and cot may then be found by means of the reciprocal key. *Before using a calculator to find function values that correspond to the radian measure of an acute angle, be sure that the calculator is in radian mode. For values corresponding to degree measure, select degree mode.*

Figure 8
In degree mode

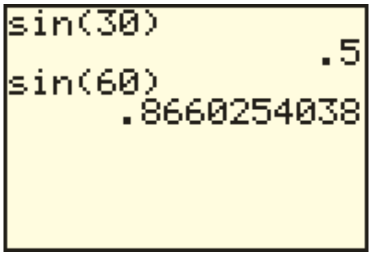
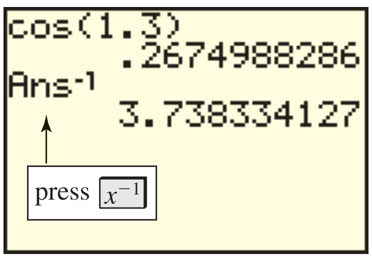


Figure 9
In radian mode



As an illustration (see Figure 8), to find sin 30° on a typical calculator, we place the calculator in degree mode and use the $\boxed{\text{SIN}}$ key to obtain $\sin 30° = 0.5$, which is the exact value. Using the same procedure for 60°, we obtain a decimal approximation to $\sqrt{3}/2$, such as

$$\sin 60° \approx 0.8660.$$

Most calculators give eight- to ten-decimal-place accuracy for such function values; throughout the text, however, we will usually round off values to four decimal places.

To find a value such as cos 1.3 (see Figure 9), where 1.3 is the radian measure of an acute angle, we place the calculator in radian mode and use the $\boxed{\text{COS}}$ key, obtaining

$$\cos 1.3 \approx 0.2675.$$

For sec 1.3, we could find cos 1.3 and then use the reciprocal key, usually labeled $\boxed{1/x}$ or $\boxed{x^{-1}}$ (as shown in Figure 9), to obtain

$$\sec 1.3 = \frac{1}{\cos 1.3} \approx 3.7383.$$

The formulas listed in the box on the next page are, without doubt, the most important identities in trigonometry, because they can be used to simplify and unify many different aspects of the subject. Since the formulas are part of the foundation for work in trigonometry, they are called the *fundamental identities*.

Three of the fundamental identities involve squares, such as $(\sin \theta)^2$ and $(\cos \theta)^2$. In general, if n is an integer different from -1, then a power such as $(\cos \theta)^n$ is written $\cos^n \theta$. The symbols $\sin^{-1} \theta$ and $\cos^{-1} \theta$ are reserved for inverse trigonometric functions, which we will discuss in Section 6.4 and treat thoroughly in the next chapter. With this agreement on notation, we have, for example,

$$\cos^2 \theta = (\cos \theta)^2 = (\cos \theta)(\cos \theta)$$
$$\tan^3 \theta = (\tan \theta)^3 = (\tan \theta)(\tan \theta)(\tan \theta)$$
$$\sec^4 \theta = (\sec \theta)^4 = (\sec \theta)(\sec \theta)(\sec \theta)(\sec \theta).$$

Let us next list all the fundamental identities and then discuss the proofs. These identities are true for every acute angle θ, and θ may take on various forms. For example, using the first Pythagorean identity with $\theta = 4\alpha$, we know that

$$\sin^2 4\alpha + \cos^2 4\alpha = 1.$$

We shall see later that these identities are also true for other angles and for real numbers.

The Fundamental Identities

(1) The reciprocal identities:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

(2) The tangent and cotangent identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

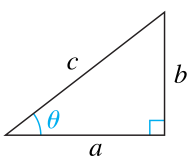
(3) The Pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta$$

PROOFS

(1) The reciprocal identities were established earlier in this section.

Figure 10



(2) To prove the tangent identity, we refer to the right triangle in Figure 10 and use definitions of trigonometric functions as follows:

$$\tan\theta = \frac{b}{a} = \frac{b/c}{a/c} = \frac{\sin\theta}{\cos\theta}$$

To verify the cotangent identity, we use a reciprocal identity and the tangent identity:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sin\theta/\cos\theta} = \frac{\cos\theta}{\sin\theta}$$

(3) The Pythagorean identities are so named because of the first step in the following proof. Referring to Figure 10, we obtain

$$b^2 + a^2 = c^2 \qquad \text{Pythagorean theorem}$$

$$\left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = \left(\frac{c}{c}\right)^2 \qquad \text{divide by } c^2$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1 \qquad \text{definitions of } \sin\theta \text{ and } \cos\theta$$

$$\sin^2\theta + \cos^2\theta = 1. \qquad \text{equivalent notation}$$

We may use this identity to verify the second Pythagorean identity as follows:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \text{divide by } \cos^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \text{equivalent equation}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + \left(\frac{\cos\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2 \qquad \text{law of exponents}$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{tangent and reciprocal identities}$$

To prove the third Pythagorean identity, $1 + \cot^2\theta = \csc^2\theta$, we could divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

We can use the fundamental identities to express each trigonometric function in terms of any other trigonometric function. Two illustrations are given in the next example.

EXAMPLE 4 Using fundamental identities

Let θ be an acute angle.

(a) Express $\sin\theta$ in terms of $\cos\theta$.

(b) Express $\tan\theta$ in terms of $\sin\theta$.

SOLUTION

(a) We may proceed as follows:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{Pythagorean identity}$$

$$\sin^2\theta = 1 - \cos^2\theta \qquad \text{isolate } \sin^2\theta$$

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta} \qquad \text{take the square root}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} \qquad \sin\theta > 0 \text{ for acute angles}$$

Later in this section (Example 12) we will consider a simplification involving a *non*-acute angle θ.

(b) If we begin with the fundamental identity

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

then all that remains is to express $\cos\theta$ in terms of $\sin\theta$. We can do this by solving $\sin^2\theta + \cos^2\theta = 1$ for $\cos\theta$, obtaining

$$\cos\theta = \sqrt{1 - \sin^2\theta} \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}.$$

(continued)

Hence,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}.$$

Fundamental identities are often used to simplify expressions involving trigonometric functions, as illustrated in the next example.

EXAMPLE 5 **Showing that an equation is an identity**

Show that the following equation is an identity by transforming the left-hand side into the right-hand side:

$$(\sec\theta + \tan\theta)(1 - \sin\theta) = \cos\theta$$

SOLUTION We begin with the left-hand side and proceed as follows:

$$\begin{aligned}
(\sec\theta + \tan\theta)(1-\sin\theta) &= \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)(1-\sin\theta) && \text{reciprocal and tangent identities}\\
&= \left(\frac{1+\sin\theta}{\cos\theta}\right)(1-\sin\theta) && \text{add fractions}\\
&= \frac{1-\sin^2\theta}{\cos\theta} && \text{multiply}\\
&= \frac{\cos^2\theta}{\cos\theta} && \sin^2\theta + \cos^2\theta = 1\\
&= \cos\theta && \text{cancel } \cos\theta
\end{aligned}$$

There are other ways to simplify the expression on the left-hand side in Example 5. We could first multiply the two factors and then simplify and combine terms. The method we employed—changing all expressions to expressions that involve only sines and cosines—is often useful. However, that technique does not always lead to the shortest possible simplification.

Hereafter, we shall use the phrase *verify an identity* instead of *show that an equation is an identity*. When verifying an identity, we often use fundamental identities and algebraic manipulations to simplify expressions, as we did in the preceding example. As with the fundamental identities, we understand that an identity that contains fractions is valid for all values of the variables such that no denominator is zero.

EXAMPLE 6 **Verifying an identity**

Verify the following identity by transforming the left-hand side into the right-hand side:

$$\frac{\tan\theta + \cos\theta}{\sin\theta} = \sec\theta + \cot\theta$$

SOLUTION We may transform the left-hand side into the right-hand side as follows:

$$\frac{\tan\theta + \cos\theta}{\sin\theta} = \frac{\tan\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} \qquad \text{divide numerator by } \sin\theta$$

$$= \frac{\left(\dfrac{\sin\theta}{\cos\theta}\right)}{\sin\theta} + \cot\theta \qquad \text{tangent and cotangent identities}$$

$$= \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta} + \cot\theta \qquad \text{rule for quotients}$$

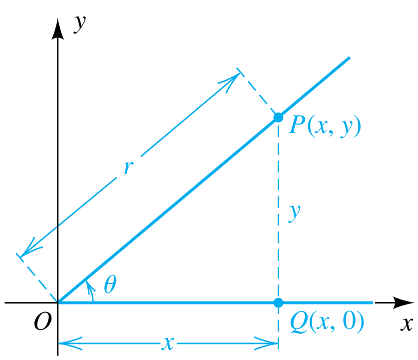
$$= \frac{1}{\cos\theta} + \cot\theta \qquad \text{cancel } \sin\theta$$

$$= \sec\theta + \cot\theta \qquad \text{reciprocal identity}$$

In Section 7.1 we will verify many other identities using methods similar to those used in Examples 5 and 6.

Since many applied problems involve angles that are not acute, it is necessary to extend the definition of the trigonometric functions. We make this extension by using the standard position of an angle θ on a rectangular coordinate system. If θ is acute, we have the situation illustrated in Figure 11, where we have chosen a point $P(x, y)$ on the terminal side of θ and where $d(O, P) = r = \sqrt{x^2 + y^2}$. Referring to triangle OQP, we have

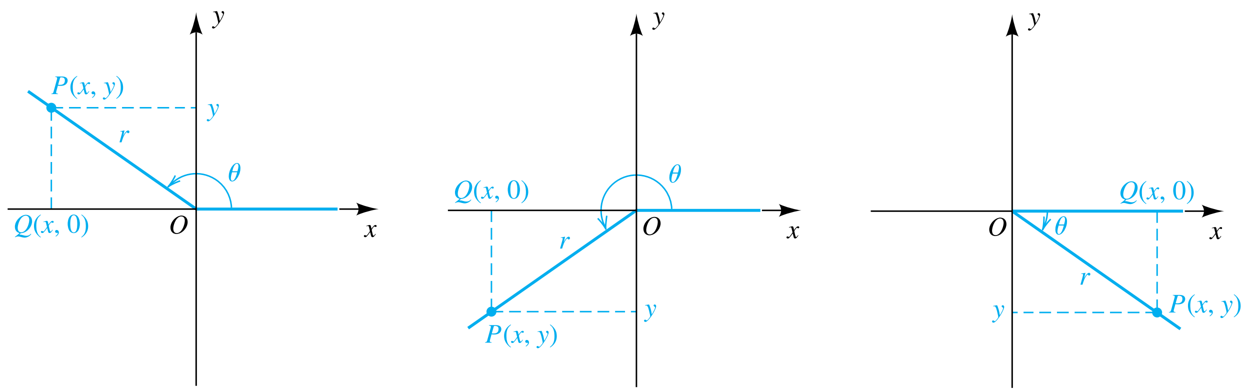
Figure 11



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}, \quad \text{and} \quad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}.$$

We now wish to consider angles of the types illustrated in Figure 12 on the next page (or any *other* angle, either positive, negative, or zero). Note that in Figure 12 the value of x or y may be negative. In each case, side QP (opp in Figure 12) has length $|y|$, side OQ (adj in Figure 12) has length $|x|$, and the hypotenuse OP has length r. We shall define the six trigonometric functions so that their values agree with those given previously whenever the angle is acute. It is understood that if a zero denominator occurs, then the corresponding function value is undefined.

Figure 12



Definition of the Trigonometric Functions of Any Angle

Let θ be an angle in standard position on a rectangular coordinate system, and let $P(x, y)$ be any point other than the origin O on the terminal side of θ. If $d(O, P) = r = \sqrt{x^2 + y^2}$, then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (\text{if } x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (\text{if } y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (\text{if } x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (\text{if } y \neq 0).$$

We can show, using similar triangles, that the formulas in this definition do not depend on the point $P(x, y)$ that is chosen on the terminal side of θ. The fundamental identities, which were established for acute angles, are also true for trigonometric functions of any angle.

The domains of the sine and cosine functions consist of all angles θ. However, $\tan\theta$ and $\sec\theta$ are undefined if $x = 0$ (that is, if the terminal side of θ is on the y-axis). Thus, the domains of the tangent and the secant functions consist of all angles *except* those of radian measure $(\pi/2) + \pi n$ for any integer n. Some special cases are $\pm\pi/2$, $\pm 3\pi/2$, and $\pm 5\pi/2$. The corresponding degree measures are $\pm 90°$, $\pm 270°$, and $\pm 450°$.

The domains of the cotangent and cosecant functions consist of all angles except those that have $y = 0$ (that is, all angles except those having terminal sides on the x-axis). These are the angles of radian measure πn (or degree measure $180° \cdot n$) for any integer n.

Our discussion of domains is summarized in the following table, where n denotes any integer.

Function		Domain
sine,	cosine	every angle θ
tangent,	secant	every angle θ except $\theta = \dfrac{\pi}{2} + \pi n = 90° + 180° \cdot n$
cotangent,	cosecant	every angle θ except $\theta = \pi n = 180° \cdot n$

For any point $P(x, y)$ in the preceding definition, $|x| \leq r$ and $|y| \leq r$ or, equivalently, $|x/r| \leq 1$ and $|y/r| \leq 1$. Thus,

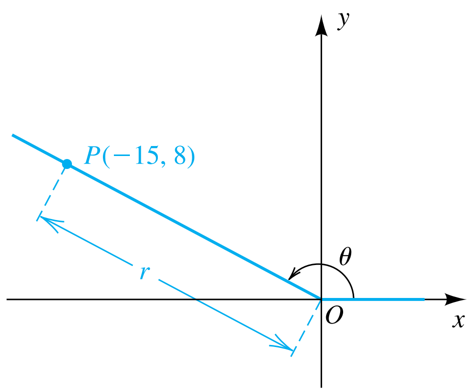
$$|\sin\theta| \leq 1, \quad |\cos\theta| \leq 1, \quad |\csc\theta| \geq 1, \quad \text{and} \quad |\sec\theta| \geq 1$$

for every θ in the domains of these functions.

EXAMPLE 7 Finding trigonometric function values of an angle in standard position

If θ is an angle in standard position on a rectangular coordinate system and if $P(-15, 8)$ is on the terminal side of θ, find the values of the six trigonometric functions of θ.

Figure 13



SOLUTION The point $P(-15, 8)$ is shown in Figure 13. Applying the definition of the trigonometric functions of any angle with $x = -15$, $y = 8$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-15)^2 + 8^2} = \sqrt{289} = 17,$$

we obtain the following:

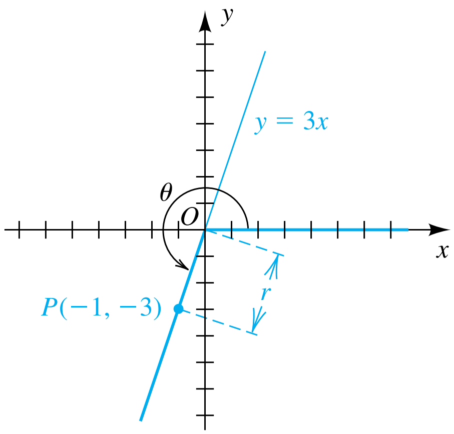
$$\sin\theta = \frac{y}{r} = \frac{8}{17} \qquad \cos\theta = \frac{x}{r} = -\frac{15}{17} \qquad \tan\theta = \frac{y}{x} = -\frac{8}{15}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{8} \qquad \sec\theta = \frac{r}{x} = -\frac{17}{15} \qquad \cot\theta = \frac{x}{y} = -\frac{15}{8}$$

EXAMPLE 8 Finding trigonometric function values of an angle in standard position

An angle θ is in standard position, and its terminal side lies in quadrant III on the line $y = 3x$. Find the values of the trigonometric functions of θ.

Figure 14



SOLUTION The graph of $y = 3x$ is sketched in Figure 14, together with the initial and terminal sides of θ. Since the terminal side of θ is in quadrant III, we begin by choosing a convenient negative value of x, say $x = -1$. Substituting in $y = 3x$ gives us $y = 3(-1) = -3$, and hence $P(-1, -3)$ is on the terminal side. Applying the definition of the trigonometric functions of any angle with

$$x = -1, \quad y = -3, \quad \text{and} \quad r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$$

gives us

$$\sin\theta = -\frac{3}{\sqrt{10}} \qquad \cos\theta = -\frac{1}{\sqrt{10}} \qquad \tan\theta = \frac{-3}{-1} = 3$$

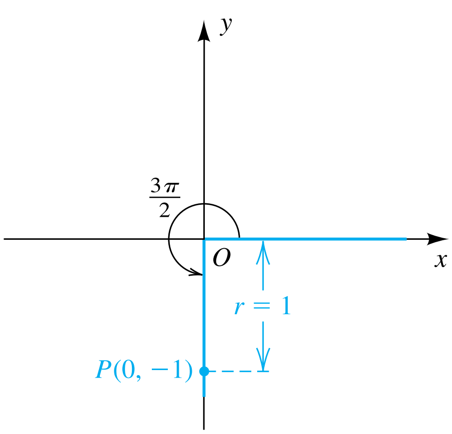
$$\csc\theta = -\frac{\sqrt{10}}{3} \qquad \sec\theta = -\frac{\sqrt{10}}{1} \qquad \cot\theta = \frac{-1}{-3} = \frac{1}{3}.$$

The definition of the trigonometric functions of any angle may be applied if θ is a quadrantal angle. The procedure is illustrated by the next example.

EXAMPLE 9 Finding trigonometric function values of a quadrantal angle

If $\theta = 3\pi/2$, find the values of the trigonometric functions of θ.

Figure 15



SOLUTION Note that $3\pi/2 = 270°$. If θ is placed in standard position, the terminal side of θ coincides with the negative y-axis, as shown in Figure 15. To apply the definition of the trigonometric functions of any angle, we may choose *any* point P on the terminal side of θ. For simplicity, we use $P(0, -1)$. In this case, $x = 0$, $y = -1$, $r = 1$, and hence

$$\sin\frac{3\pi}{2} = \frac{-1}{1} = -1 \qquad \cos\frac{3\pi}{2} = \frac{0}{1} = 0$$

$$\csc\frac{3\pi}{2} = \frac{1}{-1} = -1 \qquad \cot\frac{3\pi}{2} = \frac{0}{-1} = 0.$$

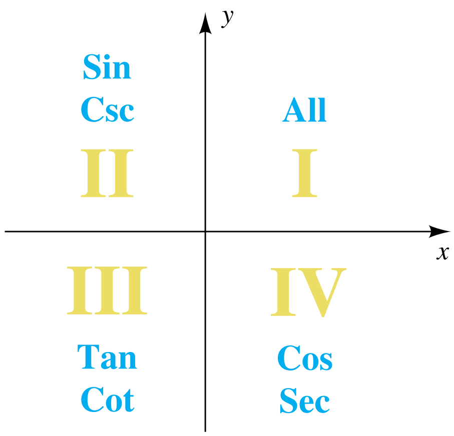
The tangent and secant functions are undefined, since the meaningless expressions $\tan\theta = (-1)/0$ and $\sec\theta = 1/0$ occur when we substitute in the appropriate formulas.

Let us determine the signs associated with values of the trigonometric functions. If θ is in quadrant II and $P(x, y)$ is a point on the terminal side, then x is negative and y is positive. Hence, $\sin\theta = y/r$ and $\csc\theta = r/y$ are positive, and the other four trigonometric functions, which all involve x, are negative. Checking the remaining quadrants in a similar fashion, we obtain the following table.

Signs of the Trigonometric Functions

Quadrant containing θ	Positive functions	Negative functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

Figure 16
Positive trigonometric functions



A mnemonic device for remembering the quadrants in which the trigonometric functions are positive is "A Smart Trig Class," which corresponds to All Sin Tan Cos.

The diagram in Figure 16 may be useful for remembering quadrants in which trigonometric functions are *positive*. If a function is not listed (such as cos in quadrant II), then that function is negative. We finish this section with three examples that require using the information in the preceding table.

EXAMPLE 10 **Finding the quadrant containing an angle**

Find the quadrant containing θ if both $\cos\theta > 0$ and $\sin\theta < 0$.

SOLUTION Referring to the table of signs or Figure 16, we see that $\cos\theta > 0$ (cosine is positive) if θ is in quadrant I or IV and that $\sin\theta < 0$ (sine is negative) if θ is in quadrant III or IV. Hence, for both conditions to be satisfied, θ must be in quadrant IV.

EXAMPLE 11 **Finding values of trigonometric functions from prescribed conditions**

If $\sin\theta = \frac{3}{5}$ and $\tan\theta < 0$, use fundamental identities to find the values of the other five trigonometric functions.

SOLUTION Since $\sin\theta = \frac{3}{5} > 0$ (positive) and $\tan\theta < 0$ (negative), θ is in quadrant II. Using the relationship $\sin^2\theta + \cos^2\theta = 1$ and the fact that $\cos\theta$ is negative in quadrant II, we have

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\left(\tfrac{3}{5}\right)^2} = -\sqrt{\tfrac{16}{25}} = -\tfrac{4}{5}.$$

Next we use the tangent identity to obtain

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{-4/5} = -\frac{3}{4}.$$

Finally, using the reciprocal identities gives us

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{3/5} = \frac{5}{3}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-4/5} = -\frac{5}{4}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-3/4} = -\frac{4}{3}.$$

EXAMPLE 12 **Using fundamental identities**

Rewrite $\sqrt{\cos^2\theta + \sin^2\theta + \cot^2\theta}$ in nonradical form without using absolute values for $\pi < \theta < 2\pi$.

SOLUTION

$$\begin{aligned} \sqrt{\cos^2\theta + \sin^2\theta + \cot^2\theta} &= \sqrt{1 + \cot^2\theta} && \cos^2\theta + \sin^2\theta = 1 \\ &= \sqrt{\csc^2\theta} && 1 + \cot^2\theta = \csc^2\theta \\ &= |\csc\theta| && \sqrt{x^2} = |x| \end{aligned}$$

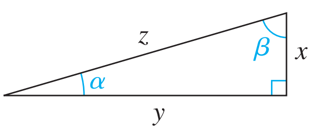
Since $\pi < \theta < 2\pi$, we know that θ is in quadrant III or IV. Thus, $\csc\theta$ is *negative,* and by the definition of absolute value, we have

$$|\csc\theta| = -\csc\theta.$$

6.2 Exercises

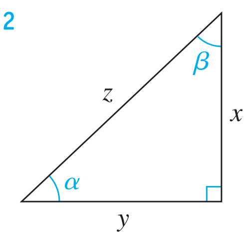
Exer. 1–2: Use common sense to match the variables and the values. (The triangles are drawn to scale, and the angles are measured in radians.)

1



(a) α	(A) 7
(b) β	(B) 0.28
(c) x	(C) 24
(d) y	(D) 1.29
(e) z	(E) 25

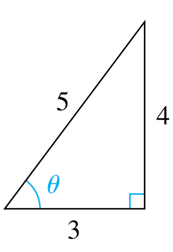
2



(a) α	(A) 23.35
(b) β	(B) 16
(c) x	(C) 17
(d) y	(D) 0.82
(e) z	(E) 0.76

Exer. 3–10: Find the values of the six trigonometric functions for the angle θ.

3



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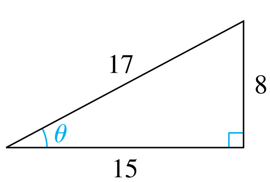

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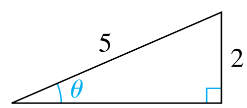

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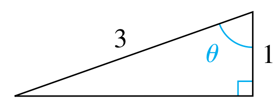

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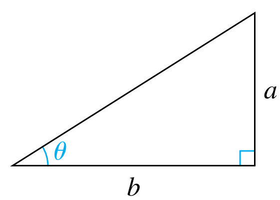

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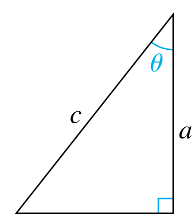

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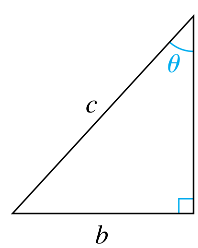

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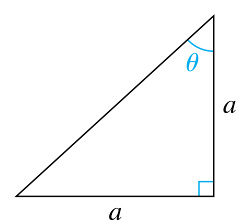
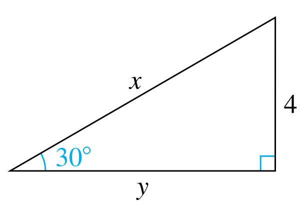


Exer. 11–16: Find the exact values of x and y.

11



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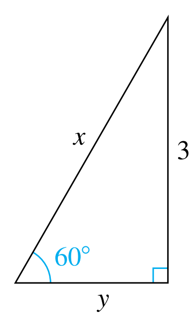

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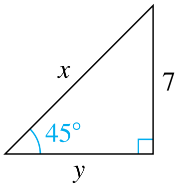

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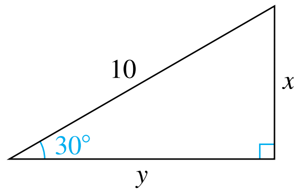

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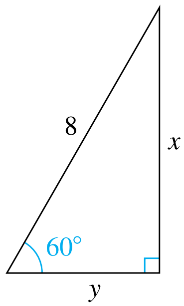

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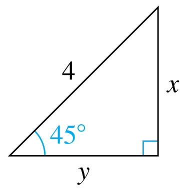


Exer. 17–22: Find the exact values of the trigonometric functions for the acute angle θ.

17 $\sin\theta = \frac{3}{5}$

18 $\cos\theta = \frac{8}{17}$

19 $\tan\theta = \frac{5}{12}$

20 $\cot\theta = \frac{7}{24}$

21 $\sec\theta = \frac{6}{5}$

22 $\csc\theta = 4$

23 **Height of a tree** A forester, 200 feet from the base of a redwood tree, observes that the angle between the ground and the top of the tree is 60°. Estimate the height of the tree.

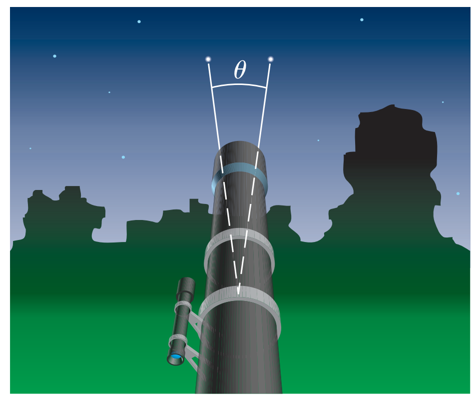
24 **Distance to Mt. Fuji** The peak of Mt. Fuji in Japan is approximately 12,400 feet high. A trigonometry student, several miles away, notes that the angle between level ground and the peak is 30°. Estimate the distance from the student to the point on level ground directly beneath the peak.

25 **Stonehenge blocks** Stonehenge in Salisbury Plains, England, was constructed using solid stone blocks weighing over 99,000 pounds each. Lifting a single stone required 550 people, who pulled the stone up a ramp inclined at an angle of 9°. Approximate the distance that a stone was moved in order to raise it to a height of 30 feet.

26 **Advertising sign height** Added in 1990 and removed in 1997, the highest advertising sign in the world was a large letter I situated at the top of the 73-story First Interstate World Center building in Los Angeles. At a distance of 200 feet from a point directly below the sign, the angle between the ground and the top of the sign was 78.87°. Approximate the height of the top of the sign.

27 **Telescope resolution** Two stars that are very close may appear to be one. The ability of a telescope to separate their images is called its resolution. The smaller the resolution, the better a telescope's ability to separate images in the sky. In a refracting telescope, resolution θ (see the figure) can be improved by using a lens with a larger diameter D. The relationship between θ in degrees and D in meters is given by $\sin\theta = 1.22\lambda/D$, where λ is the wavelength of light in meters. The largest refracting telescope in the world is at the University of Chicago. At a wavelength of $\lambda = 550 \times 10^{-9}$ meter, its resolution is 0.000 037 69°. Approximate the diameter of the lens.

Exercise 27



28 **Moon phases** The phases of the moon can be described using the phase angle θ, determined by the sun, the moon, and Earth, as shown in the figure. Because the moon orbits Earth, θ changes during the course of a month. The area of the region A of the moon, which appears illuminated to an observer on Earth, is given by $A = \frac{1}{2}\pi R^2(1 + \cos\theta)$, where $R = 1080$ mi is the radius of the moon. Approximate A for the following positions of the moon:

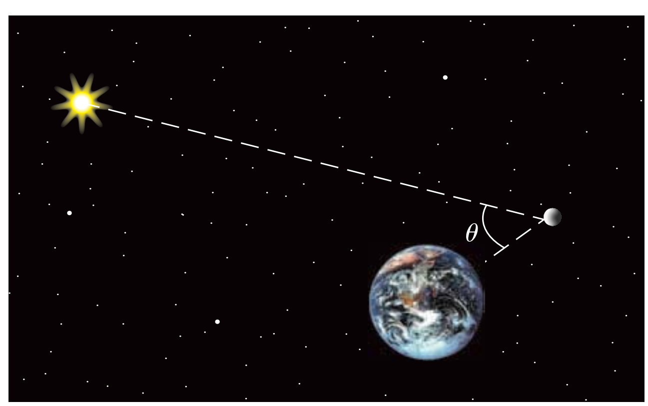
(a) $\theta = 0°$ (full moon)

(b) $\theta = 180°$ (new moon)

(c) $\theta = 90°$ (first quarter)

(d) $\theta = 103°$

Exercise 28



Exer. 29–34: Approximate to four decimal places, when appropriate.

29 (a) $\sin 42^\circ$ (b) $\cos 77^\circ$
(c) $\csc 123^\circ$ (d) $\sec(-190^\circ)$

30 (a) $\tan 282^\circ$ (b) $\cot(-81^\circ)$
(c) $\sec 202^\circ$ (d) $\sin 97^\circ$

31 (a) $\cot(\pi/13)$ (b) $\csc 1.32$
(c) $\cos(-8.54)$ (d) $\tan(3\pi/7)$

32 (a) $\sin(-0.11)$ (b) $\sec \frac{31}{27}$
(c) $\tan\left(-\frac{3}{13}\right)$ (d) $\cos 2.4\pi$

33 (a) $\sin 30^\circ$ (b) $\sin 30$
(c) $\cos \pi^\circ$ (d) $\cos \pi$

34 (a) $\sin 45^\circ$ (b) $\sin 45$
(c) $\cos(3\pi/2)^\circ$ (d) $\cos(3\pi/2)$

Exer. 35–38: Use the Pythagorean identities to write the expression as an integer.

35 (a) $\tan^2 4\beta - \sec^2 4\beta$ (b) $4\tan^2\beta - 4\sec^2\beta$

36 (a) $\csc^2 3\alpha - \cot^2 3\alpha$ (b) $3\csc^2\alpha - 3\cot^2\alpha$

37 (a) $5\sin^2\theta + 5\cos^2\theta$
(b) $5\sin^2(\theta/4) + 5\cos^2(\theta/4)$

38 (a) $7\sec^2\gamma - 7\tan^2\gamma$
(b) $7\sec^2(\gamma/3) - 7\tan^2(\gamma/3)$

Exer. 39–42: Simplify the expression.

39 $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$

40 $\dfrac{\cot^2\alpha - 4}{\cot^2\alpha - \cot\alpha - 6}$

41 $\dfrac{2 - \tan\theta}{2\csc\theta - \sec\theta}$

42 $\dfrac{\csc\theta + 1}{(1/\sin^2\theta) + \csc\theta}$

Exer. 43–48: Use fundamental identities to write the first expression in terms of the second, for any acute angle θ.

43 $\cot\theta$, $\sin\theta$
44 $\tan\theta$, $\cos\theta$
45 $\sec\theta$, $\sin\theta$
46 $\csc\theta$, $\cos\theta$
47 $\sin\theta$, $\sec\theta$
48 $\cos\theta$, $\cot\theta$

Exer. 49–70: Verify the identity by transforming the left-hand side into the right-hand side.

49 $\cos\theta\sec\theta = 1$
50 $\tan\theta\cot\theta = 1$
51 $\sin\theta\sec\theta = \tan\theta$
52 $\sin\theta\cot\theta = \cos\theta$
53 $\dfrac{\csc\theta}{\sec\theta} = \cot\theta$
54 $\cot\theta\sec\theta = \csc\theta$
55 $(1 + \cos 2\theta)(1 - \cos 2\theta) = \sin^2 2\theta$
56 $\cos^2 2\theta - \sin^2 2\theta = 2\cos^2 2\theta - 1$
57 $\cos^2\theta(\sec^2\theta - 1) = \sin^2\theta$
58 $(\tan\theta + \cot\theta)\tan\theta = \sec^2\theta$
59 $\dfrac{\sin(\theta/2)}{\csc(\theta/2)} + \dfrac{\cos(\theta/2)}{\sec(\theta/2)} = 1$
60 $1 - 2\sin^2(\theta/2) = 2\cos^2(\theta/2) - 1$
61 $(1 + \sin\theta)(1 - \sin\theta) = \dfrac{1}{\sec^2\theta}$
62 $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$
63 $\sec\theta - \cos\theta = \tan\theta\sin\theta$
64 $\dfrac{\sin\theta + \cos\theta}{\cos\theta} = 1 + \tan\theta$
65 $(\cot\theta + \csc\theta)(\tan\theta - \sin\theta) = \sec\theta - \cos\theta$
66 $\cot\theta + \tan\theta = \csc\theta\sec\theta$
67 $\sec^2 3\theta\csc^2 3\theta = \sec^2 3\theta + \csc^2 3\theta$
68 $\dfrac{1 + \cos^2 3\theta}{\sin^2 3\theta} = 2\csc^2 3\theta - 1$
69 $\log\csc\theta = -\log\sin\theta$
70 $\log\tan\theta = \log\sin\theta - \log\cos\theta$

Exer. 71–74: Find the exact values of the six trigonometric functions of θ if θ is in standard position and P is on the terminal side.

71 $P(4, -3)$
72 $P(-8, -15)$
73 $P(-2, -5)$
74 $P(-1, 2)$

Exer. 75–80: Find the exact values of the six trigonometric functions of θ if θ is in standard position and the terminal side of θ is in the specified quadrant and satisfies the given condition.

75 II; on the line $y = -4x$

76 IV; on the line $3y + 5x = 0$

77 I; on a line having slope $\frac{4}{3}$

78 III; bisects the quadrant

79 III; parallel to the line $2y - 7x + 2 = 0$

80 II; parallel to the line through $A(1, 4)$ and $B(3, -2)$

Exer. 81–82: Find the exact values of the six trigonometric functions of each angle, whenever possible.

81 (a) $90°$ (b) $0°$ (c) $7\pi/2$ (d) 3π

82 (a) $180°$ (b) $-90°$ (c) 2π (d) $5\pi/2$

Exer. 83–84: Find the quadrant containing θ if the given conditions are true.

83 (a) $\cos\theta > 0$ and $\sin\theta < 0$

(b) $\sin\theta < 0$ and $\cot\theta > 0$

(c) $\csc\theta > 0$ and $\sec\theta < 0$

(d) $\sec\theta < 0$ and $\tan\theta > 0$

84 (a) $\tan\theta < 0$ and $\cos\theta > 0$

(b) $\sec\theta > 0$ and $\tan\theta < 0$

(c) $\csc\theta > 0$ and $\cot\theta < 0$

(d) $\cos\theta < 0$ and $\csc\theta < 0$

Exer. 85–92: Use fundamental identities to find the values of the trigonometric functions for the given conditions.

85 $\tan\theta = -\frac{3}{4}$ and $\sin\theta > 0$

86 $\cot\theta = \frac{3}{4}$ and $\cos\theta < 0$

87 $\sin\theta = -\frac{5}{13}$ and $\sec\theta > 0$

88 $\cos\theta = \frac{1}{2}$ and $\sin\theta < 0$

89 $\cos\theta = -\frac{1}{3}$ and $\sin\theta < 0$

90 $\csc\theta = 5$ and $\cot\theta < 0$

91 $\sec\theta = -4$ and $\csc\theta > 0$

92 $\sin\theta = \frac{2}{5}$ and $\cos\theta < 0$

Exer. 93–98: Rewrite the expression in nonradical form without using absolute values for the indicated values of θ.

93 $\sqrt{\sec^2\theta - 1}$; $\pi/2 < \theta < \pi$

94 $\sqrt{1 + \cot^2\theta}$; $0 < \theta < \pi$

95 $\sqrt{1 + \tan^2\theta}$; $3\pi/2 < \theta < 2\pi$

96 $\sqrt{\csc^2\theta - 1}$; $3\pi/2 < \theta < 2\pi$

97 $\sqrt{\sin^2(\theta/2)}$; $2\pi < \theta < 4\pi$

98 $\sqrt{\cos^2(\theta/2)}$; $0 < \theta < \pi$

6.3 Trigonometric Functions of Real Numbers

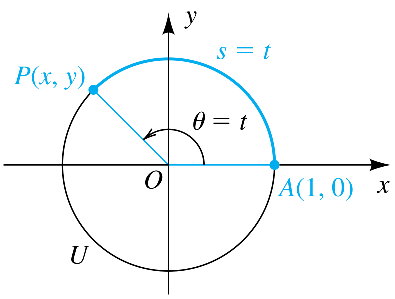
The domain of each trigonometric function we have discussed is a set of angles. In calculus and in many applications, domains of functions consist of real numbers. To regard the domain of a trigonometric function as a subset of $\mathbb{R}$, we may use the following definition.

Definition of the Trigonometric Functions of Real Numbers	The **value of a trigonometric function at a real number t** is its value at an angle of t radians, provided that value exists.

Using this definition, we may interpret a notation such as sin 2 as *either* the sine of the real number 2 *or* the sine of an angle of 2 radians. As in Section 6.2, if degree measure is used, we shall write sin 2°. With this understanding,

$$\sin 2 \neq \sin 2°.$$

Figure 1



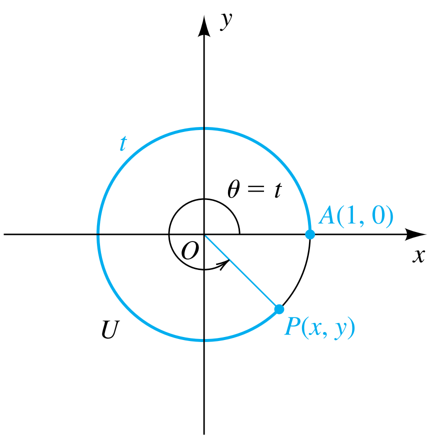
To find the values of trigonometric functions of real numbers with a calculator, we use the radian mode.

We may interpret trigonometric functions of real numbers geometrically by using a unit circle U—that is, a circle of radius 1, with center at the origin O of a rectangular coordinate plane. The circle U is the graph of the equation $x^2 + y^2 = 1$. Let t be a real number such that $0 < t < 2\pi$, and let θ denote the angle (in standard position) of radian measure t. One possibility is illustrated in Figure 1, where $P(x, y)$ is the point of intersection of the terminal side of θ and the unit circle U and where s is the length of the circular arc from $A(1, 0)$ to $P(x, y)$. Using the formula $s = r\theta$ for the length of a circular arc, with $r = 1$ and $\theta = t$, we see that

$$s = r\theta = 1(t) = t.$$

Thus, *t may be regarded either as the radian measure of the angle θ or as the length of the circular arc AP on U.*

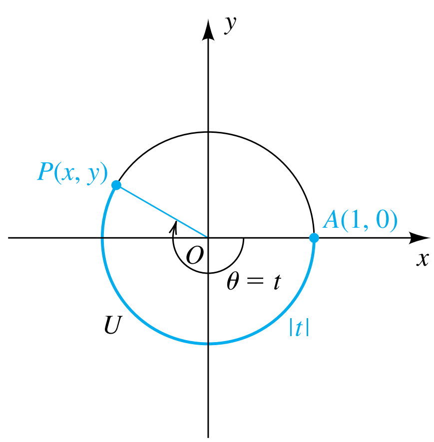
Figure 2
$\theta = t, t > 0$



Next consider *any* nonnegative real number t. If we regard the angle θ of radian measure t as having been generated by rotating the line segment OA about O in the counterclockwise direction, then t is the distance along U that A travels before reaching its final position $P(x, y)$. In Figure 2 we have illustrated a case for $t < 2\pi$; however, if $t > 2\pi$, then A may travel around U several times in a counterclockwise direction before reaching $P(x, y)$.

If $t < 0$, then the rotation of OA is in the *clockwise* direction, and the distance A travels before reaching $P(x, y)$ is $|t|$, as illustrated in Figure 3.

Figure 3
$\theta = t, t < 0$



The preceding discussion indicates how *we may associate with each real number t a unique point P(x, y) on U.* We shall call $P(x, y)$ the **point on the unit circle U that corresponds to t.** The coordinates (x, y) of P may be used to find the six trigonometric functions of t. Thus, by the definition of the trigonometric functions of real numbers together with the definition of the trigonometric functions of any angle (given in Section 6.2), we see that

$$\sin t = \sin \theta = \frac{y}{r} = \frac{y}{1} = y.$$

Using the same procedure for the remaining five trigonometric functions gives us the following formulas.

Definition of the Trigonometric Functions in Terms of a Unit Circle

If t is a real number and $P(x, y)$ is the point on the unit circle U that corresponds to t, then

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x} \quad (\text{if } x \neq 0)$$

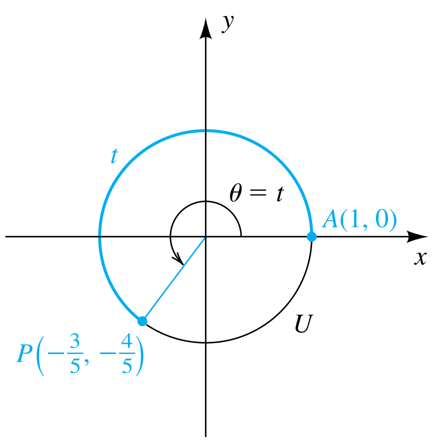
$$\csc t = \frac{1}{y} \quad (\text{if } y \neq 0) \qquad \sec t = \frac{1}{x} \quad (\text{if } x \neq 0) \qquad \cot t = \frac{x}{y} \quad (\text{if } y \neq 0).$$

The formulas in this definition express function values in terms of coordinates of a point P on a unit circle. For this reason, the trigonometric functions are sometimes referred to as the **circular functions.**

EXAMPLE 1 Finding values of the trigonometric functions

A point $P(x, y)$ on the unit circle U corresponding to a real number t is shown in Figure 4, for $\pi < t < 3\pi/2$. Find the values of the trigonometric functions at t.

Figure 4



SOLUTION Referring to Figure 4, we see that the coordinates of the point $P(x, y)$ are

$$x = -\tfrac{3}{5}, \qquad y = -\tfrac{4}{5}.$$

Using the definition of the trigonometric functions in terms of a unit circle gives us

$$\sin t = y = -\frac{4}{5} \qquad \cos t = x = -\frac{3}{5} \qquad \tan t = \frac{y}{x} = \frac{-\frac{4}{5}}{-\frac{3}{5}} = \frac{4}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{-\frac{4}{5}} = -\frac{5}{4} \quad \sec t = \frac{1}{x} = \frac{1}{-\frac{3}{5}} = -\frac{5}{3} \quad \cot t = \frac{x}{y} = \frac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}.$$

EXAMPLE 2 Finding a point on U relative to a given point

Let $P(t)$ denote the point on the unit circle U that corresponds to t for $0 \leq t < 2\pi$. If $P(t) = \left(\frac{4}{5}, \frac{3}{5}\right)$, find

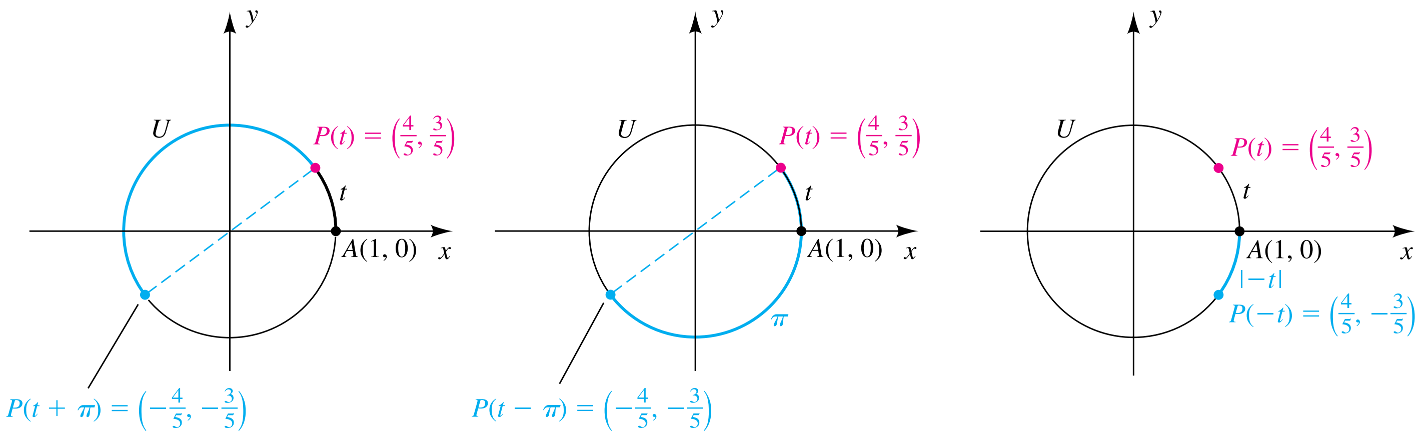
(a) $P(t + \pi)$ **(b)** $P(t - \pi)$ **(c)** $P(-t)$

SOLUTION

(a) The point $P(t)$ on U is plotted in Figure 5(a), where we have also shown the arc AP of length t. To find $P(t+\pi)$, we travel a distance π in the *counterclockwise* direction along U from $P(t)$, as indicated by the blue arc in the figure. Since π is one-half the circumference of U, this gives us the point $P(t+\pi) = \left(-\frac{4}{5}, -\frac{3}{5}\right)$ diametrically opposite $P(t)$.

Figure 5

(a) (b) (c)



(b) To find $P(t-\pi)$, we travel a distance π in the *clockwise* direction along U from $P(t)$, as indicated in Figure 5(b). This gives us $P(t-\pi) = \left(-\frac{4}{5}, -\frac{3}{5}\right)$. Note that $P(t+\pi) = P(t-\pi)$.

(c) To find $P(-t)$, we travel along U a distance $|-t|$ in the *clockwise* direction from $A(1, 0)$, as indicated in Figure 5(c). This is equivalent to reflecting $P(t)$ through the x-axis. Thus, we merely change the sign of the y-coordinate of $P(t) = \left(\frac{4}{5}, \frac{3}{5}\right)$ to obtain $P(-t) = \left(\frac{4}{5}, -\frac{3}{5}\right)$.

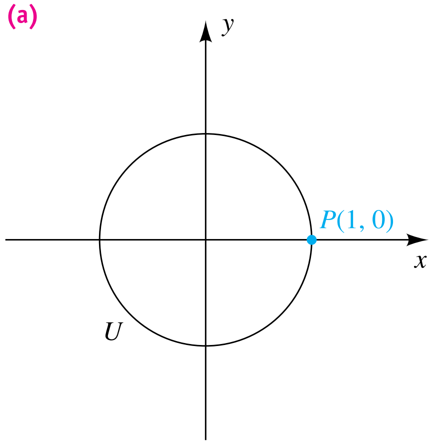
EXAMPLE 3 **Finding special values of the trigonometric functions**

Find the values of the trigonometric functions at t:

(a) $t = 0$ **(b)** $t = \dfrac{\pi}{4}$ **(c)** $t = \dfrac{\pi}{2}$

Figure 6

(a)



SOLUTION

(a) The point P on the unit circle U that corresponds to $t = 0$ has coordinates $(1, 0)$, as shown in Figure 6(a). Thus, we let $x = 1$ and $y = 0$ in the definition of the trigonometric functions in terms of a unit circle, obtaining

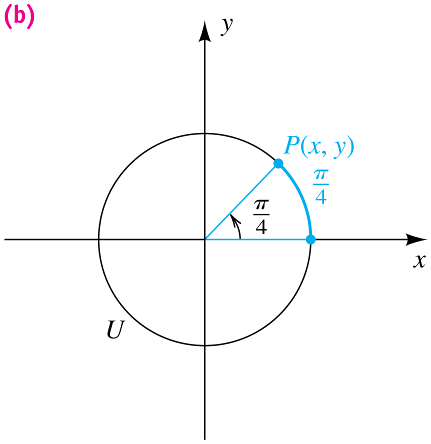
$$\sin 0 = y = 0 \qquad \cos 0 = x = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \qquad \sec 0 = \frac{1}{x} = \frac{1}{1} = 1.$$

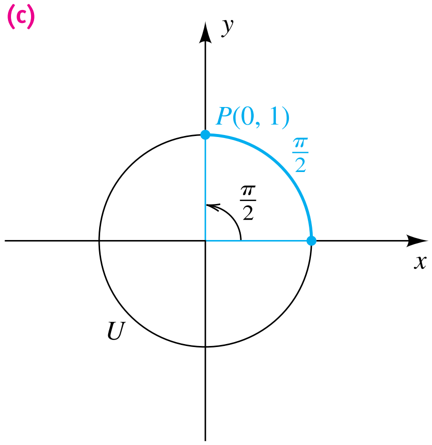
Note that $\csc 0$ and $\cot 0$ are undefined, since $y = 0$ is a denominator.

Figure 6

(b)



(c)



(b) If $t = \pi/4$, then the angle of radian measure $\pi/4$ shown in Figure 6(b) bisects the first quadrant and the point $P(x, y)$ lies on the line $y = x$. Since $P(x, y)$ is on the unit circle $x^2 + y^2 = 1$ and since $y = x$, we obtain

$$x^2 + x^2 = 1, \qquad \text{or} \qquad 2x^2 = 1.$$

Solving for x and noting that $x > 0$ gives us

$$x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Thus, P is the point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$. Letting $x = \sqrt{2}/2$ and $y = \sqrt{2}/2$ in the definition of the trigonometric functions in terms of a unit circle gives us

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

$$\csc\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad \sec\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad \cot\frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$$

(c) The point P on U that corresponds to $t = \pi/2$ has coordinates $(0, 1)$, as shown in Figure 6(c). Thus, we let $x = 0$ and $y = 1$ in the definition of the trigonometric functions in terms of a unit circle, obtaining

$$\sin\frac{\pi}{2} = 1 \qquad \cos\frac{\pi}{2} = 0 \qquad \csc\frac{\pi}{2} = \frac{1}{1} = 1 \qquad \cot\frac{\pi}{2} = \frac{0}{1} = 0.$$

The tangent and secant functions are undefined, since $x = 0$ is a denominator in each case.

A summary of the trigonometric functions of special angles appears in Appendix IV.

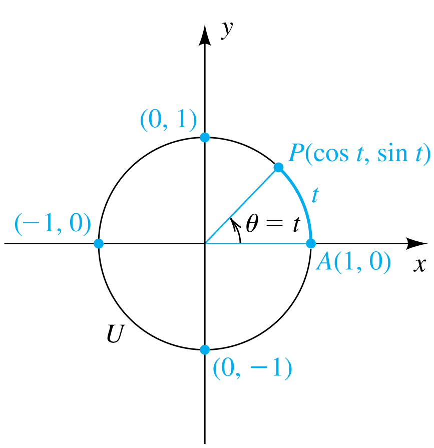
We shall use the unit circle formulation of the trigonometric functions to help obtain their graphs. If t is a real number and $P(x, y)$ is the point on the unit circle U that corresponds to t, then by the definition of the trigonometric functions in terms of a unit circle,

$$x = \cos t \qquad \text{and} \qquad y = \sin t.$$

Thus, as shown in Figure 7, we may denote $P(x, y)$ by

$$P(\cos t, \sin t).$$

Figure 7



If $t > 0$, the real number t may be interpreted either as the radian measure of the angle θ or as the length of arc AP.

If we let t increase from 0 to 2π radians, the point $P(\cos t, \sin t)$ travels around the unit circle U one time in the counterclockwise direction. By observing the variation of the x- and y-coordinates of P, we obtain the next table. The notation $0 \to \pi/2$ in the first row of the table means that t increases from 0 to $\pi/2$, and the notation $(1, 0) \to (0, 1)$ denotes the corresponding variation of $P(\cos t, \sin t)$ as it travels along U from $(1, 0)$ to $(0, 1)$. If t increases from

0 to $\pi/2$, then $\sin t$ increases from 0 to 1, which we denote by $0 \to 1$. Moreover, $\sin t$ takes on every value between 0 and 1. If t increases from $\pi/2$ to π, then $\sin t$ decreases from 1 to 0, which is denoted by $1 \to 0$. Other entries in the table may be interpreted in similar fashion.

t	$P(\cos t, \sin t)$	$\cos t$	$\sin t$
$0 \to \frac{\pi}{2}$	$(1, 0) \to (0, 1)$	$1 \to 0$	$0 \to 1$
$\frac{\pi}{2} \to \pi$	$(0, 1) \to (-1, 0)$	$0 \to -1$	$1 \to 0$
$\pi \to \frac{3\pi}{2}$	$(-1, 0) \to (0, -1)$	$-1 \to 0$	$0 \to -1$
$\frac{3\pi}{2} \to 2\pi$	$(0, -1) \to (1, 0)$	$0 \to 1$	$-1 \to 0$

If t increases from 2π to 4π, the point $P(\cos t, \sin t)$ in Figure 7 traces the unit circle U again and the patterns for $\sin t$ and $\cos t$ are repeated—that is,

$$\sin (t + 2\pi) = \sin t \qquad \text{and} \qquad \cos (t + 2\pi) = \cos t$$

for every t in the interval $[0, 2\pi]$. The same is true if t increases from 4π to 6π, from 6π to 8π, and so on. In general, we have the following theorem.

Theorem on Repeated Function Values for sin and cos

If n is any integer, then

$$\sin (t + 2\pi n) = \sin t \qquad \text{and} \qquad \cos (t + 2\pi n) = \cos t.$$

The repetitive variation of the sine and cosine functions is *periodic* in the sense of the following definition.

Definition of Periodic Function

A function f is **periodic** if there exists a positive real number k such that

$$f(t + k) = f(t)$$

for every t in the domain of f. The least such positive real number k, if it exists, is the **period** of f.

You already have a common-sense grasp of the concept of the period of a function. For example, if you were asked on a Monday "What day of the week will it be in 15 days?" your response would be "Tuesday" due to your understanding that the days of the week repeat every 7 days and 15 is one day more

x	$y = \sin x$
0	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$
$\frac{\pi}{2}$	1
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$
π	0
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.7$
$\frac{3\pi}{2}$	-1
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.7$
2π	0

than two complete periods of 7 days. From the discussion preceding the previous theorem, we see that the period of the sine and cosine functions is 2π.

We may now readily obtain the graphs of the sine and cosine functions. Since we wish to sketch these graphs on an xy-plane, let us replace the variable t by x and consider the equations

$$y = \sin x \qquad \text{and} \qquad y = \cos x.$$

We may think of x as the radian measure of any angle; however, in calculus, x is usually regarded as a real number. These are equivalent points of view, since the sine (or cosine) of an angle of x radians is the same as the sine (or cosine) of the real number x. The variable y denotes the function value that corresponds to x.

The table in the margin lists coordinates of several points on the graph of $y = \sin x$ for $0 \le x \le 2\pi$. Additional points can be determined using results on special angles, such as

$$\sin(\pi/6) = 1/2 \qquad \text{and} \qquad \sin(\pi/3) = \sqrt{3}/2 \approx 0.8660.$$

To sketch the graph for $0 \le x \le 2\pi$, we plot the points given by the table and remember that $\sin x$ increases on $[0, \pi/2]$, decreases on $[\pi/2, \pi]$ and $[\pi, 3\pi/2]$, and increases on $[3\pi/2, 2\pi]$. This gives us the sketch in Figure 8. Since the sine function is periodic, the pattern shown in Figure 8 is repeated to the right and to the left, in intervals of length 2π. This gives us the sketch in Figure 9.

Figure 8

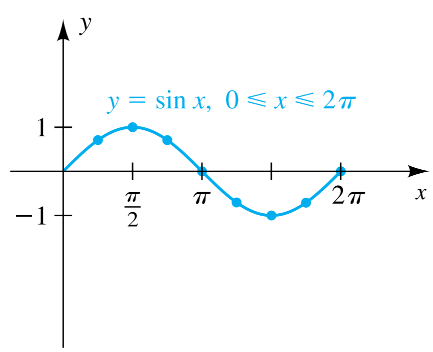
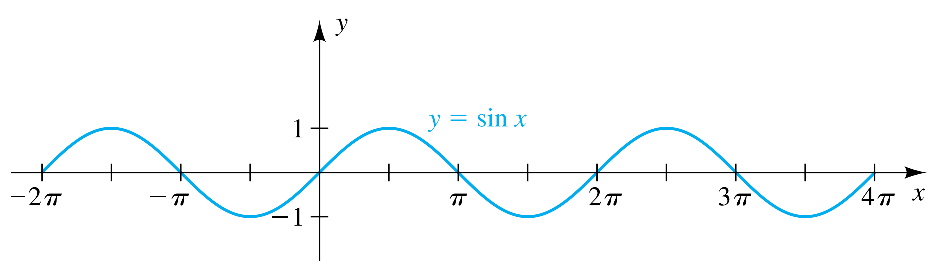


Figure 9



x	$y = \cos x$
0	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.7$
π	-1
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.7$
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$
2π	1

We can use the same procedure to sketch the graph of $y = \cos x$. The table in the margin lists coordinates of several points on the graph for $0 \leq x \leq 2\pi$. Plotting these points leads to the part of the graph shown in Figure 10. Repeating this pattern to the right and to the left, in intervals of length 2π, we obtain the sketch in Figure 11.

Figure 10

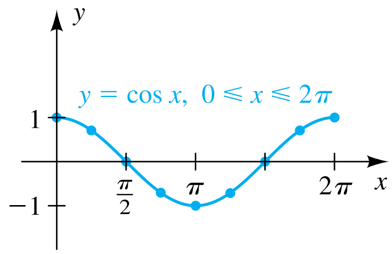
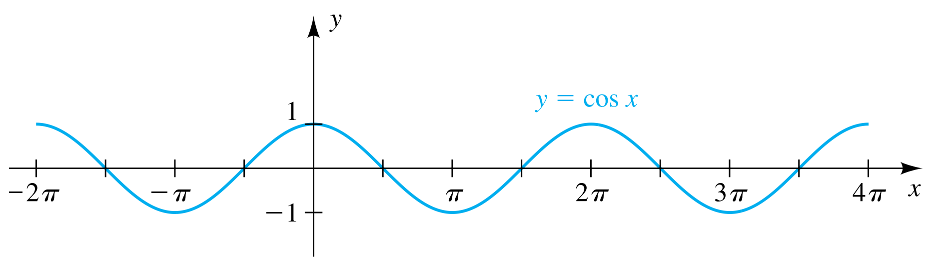


Figure 11



The part of the graph of the sine or cosine function corresponding to $0 \leq x \leq 2\pi$ is one **cycle.** We sometimes refer to a cycle as a **sine wave** or a **cosine wave.**

The range of the sine and cosine functions consists of all real numbers in the closed interval $[-1, 1]$. Since $\csc x = 1/\sin x$ and $\sec x = 1/\cos x$, it follows that the range of the cosecant and secant functions consists of all real numbers having absolute value greater than or equal to 1.

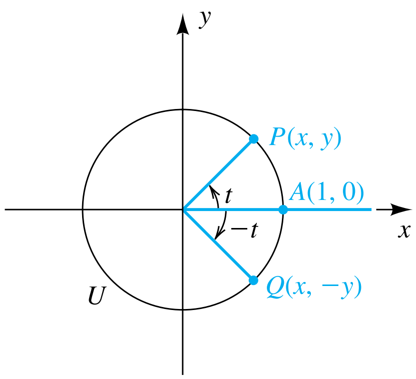
As we shall see, the range of the tangent and cotangent functions consists of all real numbers.

Before discussing graphs of the other trigonometric functions, let us establish formulas that involve functions of $-t$ for any t. Since a minus sign is involved, we call them *formulas for negatives.*

Formulas for Negatives

$$\sin(-t) = -\sin t \qquad \cos(-t) = \cos t \qquad \tan(-t) = -\tan t$$
$$\csc(-t) = -\csc t \qquad \sec(-t) = \sec t \qquad \cot(-t) = -\cot t$$

Figure 12



PROOFS Consider the unit circle U in Figure 12. As t increases from 0 to 2π, the point $P(x, y)$ traces the unit circle U once in the counterclockwise direction and the point $Q(x, -y)$, corresponding to $-t$, traces U once in the clockwise direction. Applying the definition of the trigonometric functions of any angle (with $r = 1$), we have

$$\sin(-t) = -y = -\sin t$$

$$\cos(-t) = x = \cos t$$

$$\tan(-t) = \frac{-y}{x} = -\frac{y}{x} = -\tan t.$$

The proofs of the remaining three formulas are similar.

In the following illustration, formulas for negatives are used to find an exact value for each trigonometric function.

ILLUSTRATION Use of Formulas for Negatives

- $\sin(-45°) = -\sin 45° = -\dfrac{\sqrt{2}}{2}$
- $\cos(-30°) = \cos 30° = \dfrac{\sqrt{3}}{2}$
- $\tan\left(-\dfrac{\pi}{3}\right) = -\tan\left(\dfrac{\pi}{3}\right) = -\sqrt{3}$
- $\csc(-30°) = -\csc 30° = -2$
- $\sec(-60°) = \sec 60° = 2$
- $\cot\left(-\dfrac{\pi}{4}\right) = -\cot\left(\dfrac{\pi}{4}\right) = -1$

We shall next use formulas for negatives to verify a trigonometric identity.

EXAMPLE 4 Using formulas for negatives to verify an identity

Verify the following identity by transforming the left-hand side into the right-hand side:

$$\sin(-x)\tan(-x) + \cos(-x) = \sec x$$

SOLUTION We may proceed as follows:

$$
\begin{aligned}
\sin(-x)\tan(-x) + \cos(-x) &= (-\sin x)(-\tan x) + \cos x && \text{formulas for negatives} \\
&= \sin x \, \frac{\sin x}{\cos x} + \cos x && \text{tangent identity} \\
&= \frac{\sin^2 x}{\cos x} + \cos x && \text{multiply} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x} && \text{add terms} \\
&= \frac{1}{\cos x} && \text{Pythagorean identity} \\
&= \sec x && \text{reciprocal identity}
\end{aligned}
$$

We may use the formulas for negatives to prove the following theorem.

Theorem on Even and Odd Trigonometric Functions	**(1)** The cosine and secant functions are even. **(2)** The sine, tangent, cotangent, and cosecant functions are odd.

PROOFS We shall prove the theorem for the cosine and sine functions. If $f(x) = \cos x$, then

$$f(-x) = \cos(-x) = \cos x = f(x),$$

which means that the cosine function is even.

If $f(x) = \sin x$, then

$$f(-x) = \sin(-x) = -\sin x = -f(x).$$

Thus, the sine function is odd.

Since the sine function is odd, its graph is symmetric with respect to the origin (see Figure 13). Since the cosine function is even, its graph is symmetric with respect to the y-axis (see Figure 14).

Figure 13 sine is odd

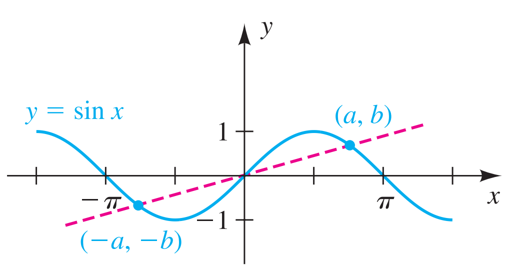
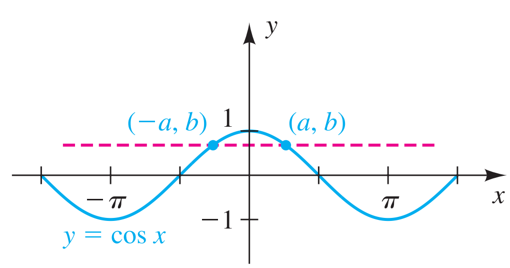
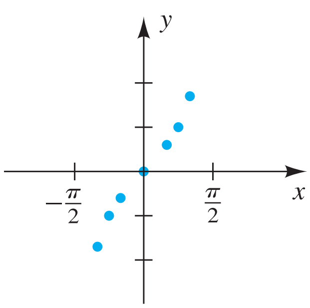


Figure 14 cosine is even



x	$y = \tan x$
$-\dfrac{\pi}{3}$	$-\sqrt{3} \approx -1.7$
$-\dfrac{\pi}{4}$	-1
$-\dfrac{\pi}{6}$	$-\dfrac{\sqrt{3}}{3} \approx -0.6$
0	0
$\dfrac{\pi}{6}$	$\dfrac{\sqrt{3}}{3} \approx 0.6$
$\dfrac{\pi}{4}$	1
$\dfrac{\pi}{3}$	$\sqrt{3} \approx 1.7$

Figure 15



By the preceding theorem, the tangent function is odd, and hence the graph of $y = \tan x$ is symmetric with respect to the origin. The table in the margin lists some points on the graph if $-\pi/2 < x < \pi/2$. The corresponding points are plotted in Figure 15. The values of $\tan x$ near $x = \pi/2$ require special attention. If we consider $\tan x = \sin x/\cos x$, then as x increases toward $\pi/2$, the numerator $\sin x$ approaches 1 and the denominator $\cos x$ approaches 0. Consequently, $\tan x$ takes on large positive values. Following are some approximations of $\tan x$ for x close to $\pi/2 \approx 1.5708$:

$$\begin{aligned}
\tan 1.57000 &\approx 1{,}255.8\\
\tan 1.57030 &\approx 2{,}014.8\\
\tan 1.57060 &\approx 5{,}093.5\\
\tan 1.57070 &\approx 10{,}381.3\\
\tan 1.57079 &\approx 158{,}057.9
\end{aligned}$$

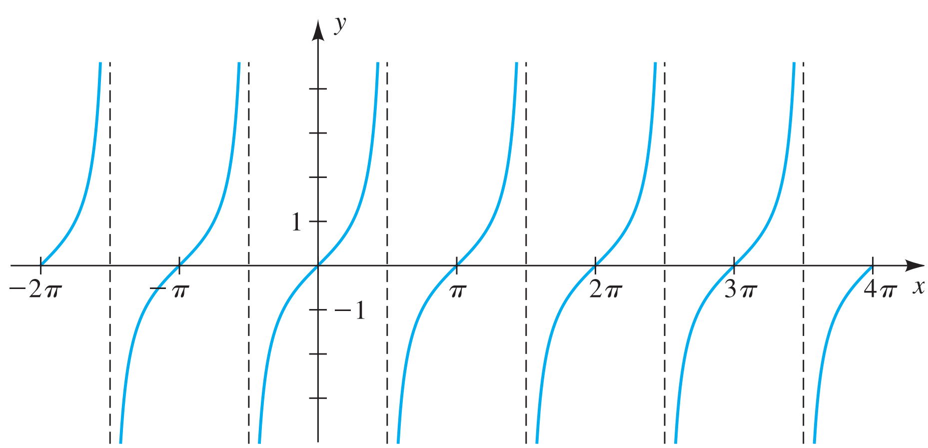
Notice how rapidly $\tan x$ increases as x approaches $\pi/2$. We say that $\tan x$ *increases without bound as x approaches $\pi/2$ through values less than $\pi/2$.* Similarly, if x approaches $-\pi/2$ through values *greater* than $-\pi/2$, then $\tan x$ *decreases without bound.* We may denote this variation using the notation introduced for rational functions in Section 4.5:

$$\text{as} \quad x \to \frac{\pi}{2}^-, \quad \tan x \to \infty$$

$$\text{as} \quad x \to -\frac{\pi}{2}^+, \quad \tan x \to -\infty$$

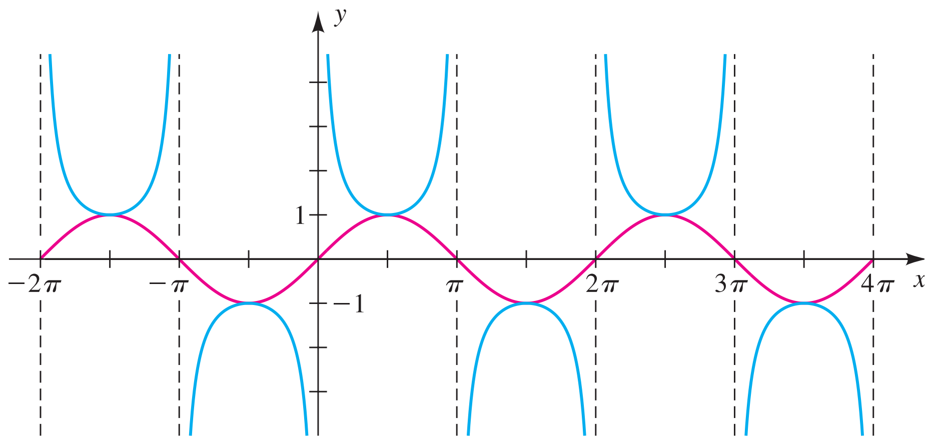
This variation of $\tan x$ in the open interval $(-\pi/2, \pi/2)$ is illustrated in Figure 16. This portion of the graph is called one **branch** of the tangent. The lines $x = -\pi/2$ and $x = \pi/2$ are vertical asymptotes for the graph. The same pattern is repeated in the open intervals $(-3\pi/2, -\pi/2)$, $(\pi/2, 3\pi/2)$, and $(3\pi/2, 5\pi/2)$ and in similar intervals of length π, as shown in the figure. Thus, *the tangent function is periodic with period π.*

Figure 16 $y = \tan x$



We may use the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ to help sketch the graphs of the remaining three trigonometric functions. For example, since $\csc x = 1/\sin x$, we may find the y-coordinate of a point on the graph of the cosecant function by taking the reciprocal of the corresponding y-coordinate on the sine graph for every value of x except $x = \pi n$ for any integer n. (If $x = \pi n$, $\sin x = 0$, and hence $1/\sin x$ is undefined.) As an aid to sketching the graph of the cosecant function, it is convenient to sketch the graph of the sine function (shown in red in Figure 17) and then take reciprocals to obtain points on the cosecant graph.

Figure 17 $y = \csc x$, $y = \sin x$



Notice the manner in which the cosecant function increases or decreases without bound as x approaches πn for any integer n. The graph has vertical asymptotes $x = \pi n$, as indicated in the figure. There is one **upper branch** of the cosecant on the interval $(0, \pi)$ and one **lower branch** on the interval $(\pi, 2\pi)$—together they compose one *cycle* of the cosecant.

Since $\sec x = 1/\cos x$ and $\cot x = 1/\tan x$, we may obtain the graphs of the secant and cotangent functions by taking reciprocals of y-coordinates of points on the graphs of the cosine and tangent functions, as illustrated in Figures 18 and 19.

Figure 18 $y = \sec x$, $y = \cos x$

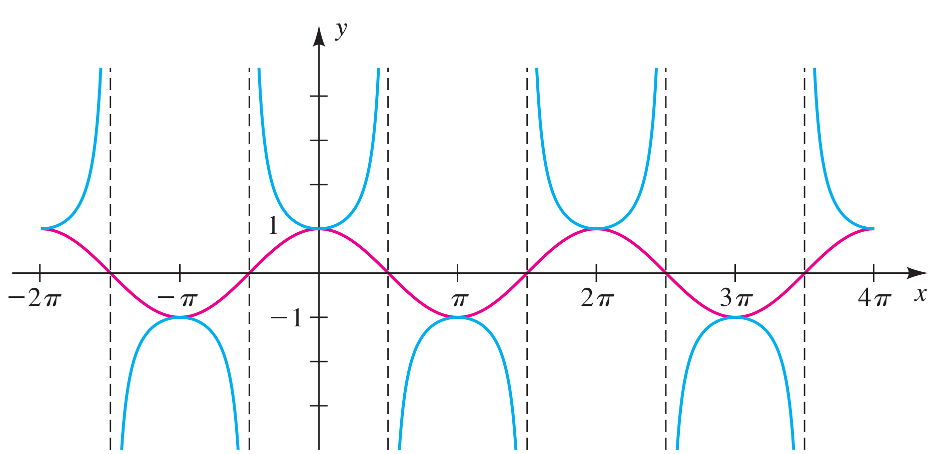
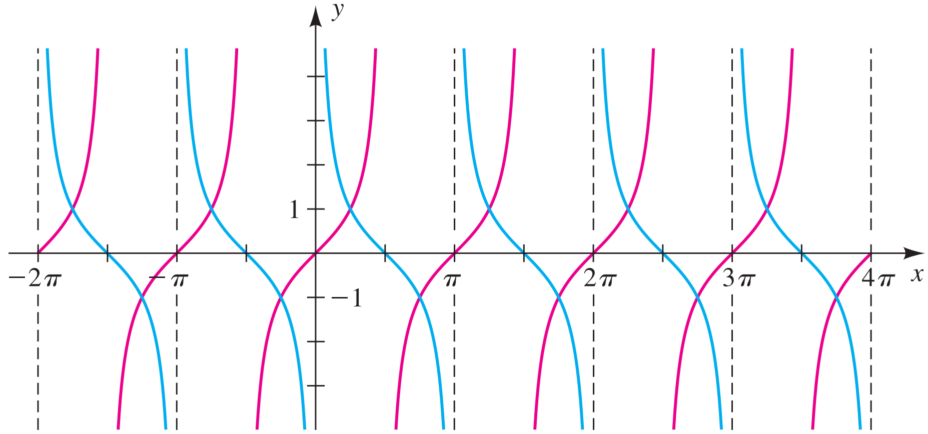


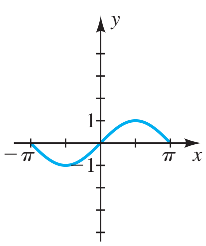
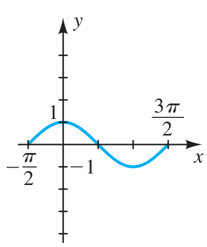
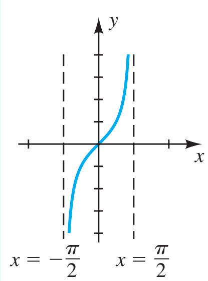
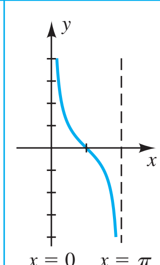
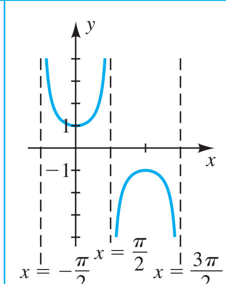
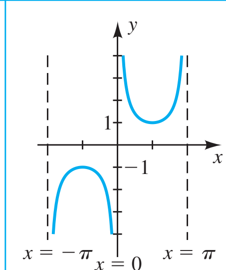
Figure 19 $y = \cot x$, $y = \tan x$



A graphical summary of the six trigonometric functions and their inverses (discussed in Section 7.6) appears in Appendix III.

We have considered many properties of the six trigonometric functions of x, where x is a real number or the radian measure of an angle. The following chart contains a summary of important features of these functions (n denotes an arbitrary integer).

Summary of Features of the Trigonometric Functions and Their Graphs

Feature	$y = \sin x$	$y = \cos x$	$y = \tan x$	$y = \cot x$	$y = \sec x$	$y = \csc x$
Graph (one period)						
Domain	$\mathbb{R}$	$\mathbb{R}$	$x \neq \frac{\pi}{2} + \pi n$	$x \neq \pi n$	$x \neq \frac{\pi}{2} + \pi n$	$x \neq \pi n$
Vertical asymptotes	none	none	$x = \frac{\pi}{2} + \pi n$	$x = \pi n$	$x = \frac{\pi}{2} + \pi n$	$x = \pi n$
Range	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$	$\mathbb{R}$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
x-intercepts	πn	$\frac{\pi}{2} + \pi n$	πn	$\frac{\pi}{2} + \pi n$	none	none
y-intercept	0	1	0	none	1	none
Period	2π	2π	π	π	2π	2π
Even or odd	odd	even	odd	odd	even	odd
Symmetry	origin	y-axis	origin	origin	y-axis	origin

EXAMPLE 5 **Investigating the variation of csc x**

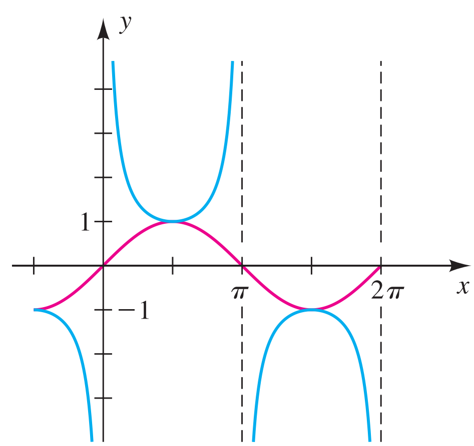
Investigate the variation of csc x as

$$x \to \pi^-, \quad x \to \pi^+, \quad x \to \frac{\pi}{2}^-, \quad \text{and} \quad x \to \frac{\pi}{6}^+.$$

SOLUTION Referring to the graph of $y = \csc x$ in Figure 20 and using our knowledge of the special values of the sine and cosecant functions, we obtain the following:

as	$x \to \pi^-$,	$\sin x \to 0$ (through positive values)	and	$\csc x \to \infty$
as	$x \to \pi^+$,	$\sin x \to 0$ (through negative values)	and	$\csc x \to -\infty$
as	$x \to \frac{\pi}{2}^-$,	$\sin x \to 1$	and	$\csc x \to 1$
as	$x \to \frac{\pi}{6}^+$,	$\sin x \to \frac{1}{2}$	and	$\csc x \to 2$

Figure 20
$y = \csc x$, $y = \sin x$



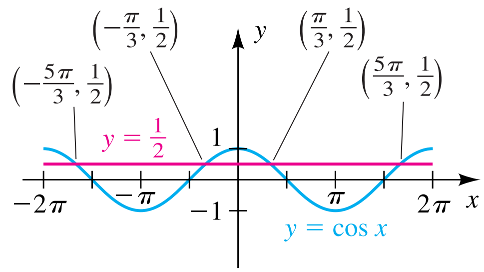
EXAMPLE 6 **Solving equations and inequalities that involve a trigonometric function**

Find all values of x in the interval $[-2\pi, 2\pi]$ such that

(a) $\cos x = \frac{1}{2}$ **(b)** $\cos x > \frac{1}{2}$ **(c)** $\cos x < \frac{1}{2}$

SOLUTION This problem can be easily solved by referring to the graphs of $y = \cos x$ and $y = \frac{1}{2}$, sketched on the same xy-plane in Figure 21 for $-2\pi \le x \le 2\pi$.

Figure 21



(a) The values of x such that $\cos x = \frac{1}{2}$ are the x-coordinates of the points at which the graphs intersect. Recall that $x = \pi/3$ satisfies the equation. By symmetry, $x = -\pi/3$ is another solution of $\cos x = \frac{1}{2}$. Since the cosine function has period 2π, the other values of x in $[-2\pi, 2\pi]$ such that $\cos x = \frac{1}{2}$ are

$$-\frac{\pi}{3} + 2\pi = \frac{5\pi}{3} \quad \text{and} \quad \frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}.$$

(b) The values of x such that $\cos x > \frac{1}{2}$ can be found by determining where the graph of $y = \cos x$ in Figure 21 lies *above* the line $y = \frac{1}{2}$. This gives us the x-intervals

$$\left[-2\pi, -\frac{5\pi}{3}\right), \quad \left(-\frac{\pi}{3}, \frac{\pi}{3}\right), \quad \text{and} \quad \left(\frac{5\pi}{3}, 2\pi\right].$$

(c) To solve $\cos x < \frac{1}{2}$, we again refer to Figure 21 and note where the graph of $y = \cos x$ lies *below* the line $y = \frac{1}{2}$. This gives us the x-intervals

$$\left(-\frac{5\pi}{3}, -\frac{\pi}{3}\right) \quad \text{and} \quad \left(\frac{\pi}{3}, \frac{5\pi}{3}\right).$$

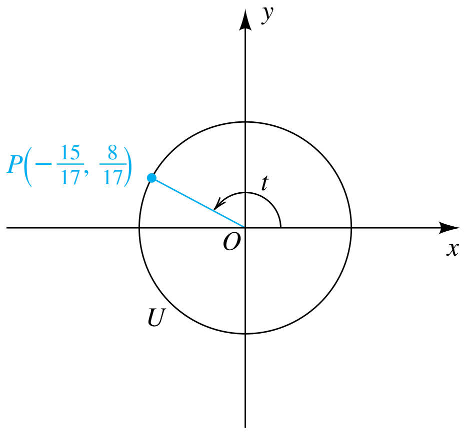
Another method of solving $\cos x < \frac{1}{2}$ is to note that the solutions are the open subintervals of $[-2\pi, 2\pi]$ that are *not* included in the intervals obtained in part (b).

We have now discussed two different approaches to the trigonometric functions. The development in terms of angles and ratios, introduced in Section 6.2, has many applications in the sciences and engineering. The definition in terms of a unit circle, considered in this section, emphasizes the fact that the trigonometric functions have domains consisting of real numbers. Such functions are the building blocks for calculus. In addition, the unit circle approach is useful for discussing graphs and deriving trigonometric identities. You should work to become proficient in the use of both formulations of the trigonometric functions, since each will reinforce the other and thus facilitate your mastery of more advanced aspects of trigonometry.

6.3 Exercises

Exer. 1–4: A point $P(x, y)$ is shown on the unit circle U corresponding to a real number t. Find the values of the trigonometric functions at t.

1



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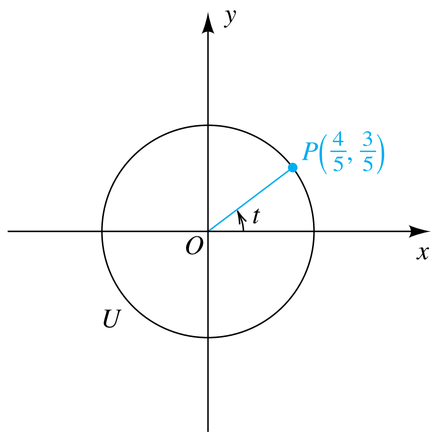

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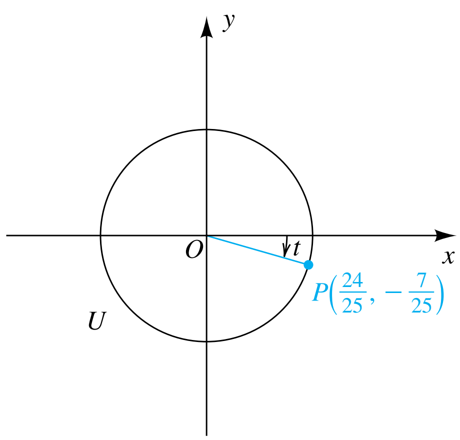

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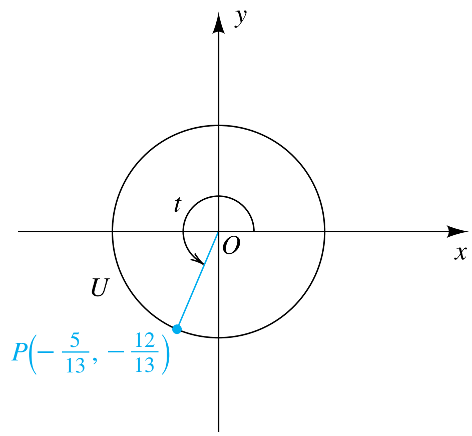


Exer. 5–8: Let $P(t)$ be the point on the unit circle U that corresponds to t. If $P(t)$ has the given rectangular coordinates, find
(a) $P(t + \pi)$ (b) $P(t - \pi)$ (c) $P(-t)$ (d) $P(-t - \pi)$

5 $\left(\frac{3}{5}, \frac{4}{5}\right)$ 6 $\left(-\frac{8}{17}, \frac{15}{17}\right)$

7 $\left(-\frac{12}{13}, -\frac{5}{13}\right)$ 8 $\left(\frac{7}{25}, -\frac{24}{25}\right)$

Exer. 9–16: Let P be the point on the unit circle U that corresponds to t. Find the coordinates of P and the exact values of the trigonometric functions of t, whenever possible.

9 (a) 2π (b) -3π

10 (a) $-\pi$ (b) 6π

11 (a) $3\pi/2$ (b) $-7\pi/2$

12 (a) $5\pi/2$ (b) $-\pi/2$

13 (a) $9\pi/4$ (b) $-5\pi/4$

14 (a) $3\pi/4$ (b) $-7\pi/4$

15 (a) $5\pi/4$ (b) $-\pi/4$

16 (a) $7\pi/4$ (b) $-3\pi/4$

Exer. 17–20: Use a formula for negatives to find the exact value.

17 (a) $\sin(-90^\circ)$ (b) $\cos\left(-\frac{3\pi}{4}\right)$ (c) $\tan(-45^\circ)$

18 (a) $\sin\left(-\frac{3\pi}{2}\right)$ (b) $\cos(-225^\circ)$ (c) $\tan(-\pi)$

19 (a) $\cot\left(-\frac{3\pi}{4}\right)$ (b) $\sec(-180^\circ)$ (c) $\csc\left(-\frac{3\pi}{2}\right)$

20 (a) $\cot(-225^\circ)$ (b) $\sec\left(-\dfrac{\pi}{4}\right)$ (c) $\csc(-45^\circ)$

Exer. 21–26: Verify the identity by transforming the left-hand side into the right-hand side.

21 $\sin(-x)\sec(-x) = -\tan x$

22 $\csc(-x)\cos(-x) = -\cot x$

23 $\dfrac{\cot(-x)}{\csc(-x)} = \cos x$

24 $\dfrac{\sec(-x)}{\tan(-x)} = -\csc x$

25 $\dfrac{1}{\cos(-x)} - \tan(-x)\sin(-x) = \cos x$

26 $\cot(-x)\cos(-x) + \sin(-x) = -\csc x$

Exer. 27–38: Complete the statement by referring to a graph of a trigonometric function.

27 (a) As $x \to 0^+$, $\sin x \to$ ___

(b) As $x \to (-\pi/2)^-$, $\sin x \to$ ___

28 (a) As $x \to \pi^+$, $\sin x \to$ ___

(b) As $x \to (\pi/6)^-$, $\sin x \to$ ___

29 (a) As $x \to (\pi/4)^+$, $\cos x \to$ ___

(b) As $x \to \pi^-$, $\cos x \to$ ___

30 (a) As $x \to 0^+$, $\cos x \to$ ___

(b) As $x \to (-\pi/3)^-$, $\cos x \to$ ___

31 (a) As $x \to (\pi/4)^+$, $\tan x \to$ ___

(b) As $x \to (\pi/2)^+$, $\tan x \to$ ___

32 (a) As $x \to 0^+$, $\tan x \to$ ___

(b) As $x \to (-\pi/2)^-$, $\tan x \to$ ___

33 (a) As $x \to (-\pi/4)^-$, $\cot x \to$ ___

(b) As $x \to 0^+$, $\cot x \to$ ___

34 (a) As $x \to (\pi/6)^+$, $\cot x \to$ ___

(b) As $x \to \pi^-$, $\cot x \to$ ___

35 (a) As $x \to (\pi/2)^-$, $\sec x \to$ ___

(b) As $x \to (\pi/4)^+$, $\sec x \to$ ___

36 (a) As $x \to (\pi/2)^+$, $\sec x \to$ ___

(b) As $x \to 0^-$, $\sec x \to$ ___

37 (a) As $x \to 0^-$, $\csc x \to$ ___

(b) As $x \to (\pi/2)^+$, $\csc x \to$ ___

38 (a) As $x \to \pi^+$, $\csc x \to$ ___

(b) As $x \to (\pi/4)^-$, $\csc x \to$ ___

Exer. 39–46: Refer to the graph of $y = \sin x$ or $y = \cos x$ to find the exact values of x in the interval $[0, 4\pi]$ that satisfy the equation.

39 $\sin x = -1$

40 $\sin x = 1$

41 $\sin x = \frac{1}{2}$

42 $\sin x = -\sqrt{2}/2$

43 $\cos x = 1$

44 $\cos x = -1$

45 $\cos x = \sqrt{2}/2$

46 $\cos x = -\frac{1}{2}$

Exer. 47–50: Refer to the graph of $y = \tan x$ to find the exact values of x in the interval $(-\pi/2, 3\pi/2)$ that satisfy the equation.

47 $\tan x = 1$

48 $\tan x = \sqrt{3}$

49 $\tan x = 0$

50 $\tan x = -1/\sqrt{3}$

Exer. 51–54: Refer to the graph of the equation on the specified interval. Find all values of x such that for the real number a, (a) $y = a$, (b) $y > a$, and (c) $y < a$.

51 $y = \sin x$; $[-2\pi, 2\pi]$; $a = \frac{1}{2}$

52 $y = \cos x$; $[0, 4\pi]$; $a = \sqrt{3}/2$

53 $y = \cos x$; $[-2\pi, 2\pi]$; $a = -\frac{1}{2}$

54 $y = \sin x$; $[0, 4\pi]$; $a = -\sqrt{2}/2$

Exer. 55–62: Use the graph of a trigonometric function to sketch the graph of the equation without plotting points.

55 $y = 2 + \sin x$

56 $y = 3 + \cos x$

57 $y = \cos x - 2$

58 $y = \sin x - 1$

59 $y = 1 + \tan x$

60 $y = \cot x - 1$

61 $y = \sec x - 2$

62 $y = 1 + \csc x$

Exer. 63–66: Find the intervals between -2π and 2π on which the given function is (a) increasing or (b) decreasing.

63 secant

64 cosecant

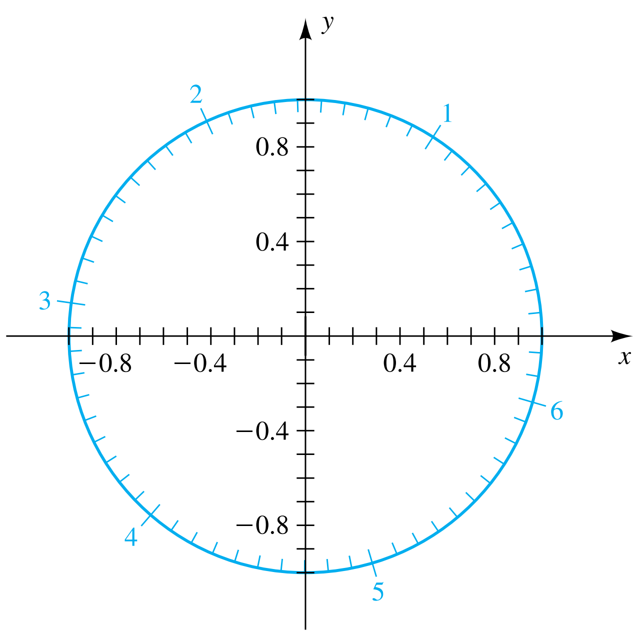
65 tangent

66 cotangent

67 Practice sketching the graph of the sine function, taking different units of length on the horizontal and vertical axes. Practice sketching graphs of the cosine and tangent functions in the same manner. Continue this practice until you reach the stage at which, if you were awakened from a sound sleep in the middle of the night and asked to sketch one of these graphs, you could do so in less than thirty seconds.

68 Work Exercise 67 for the cosecant, secant, and cotangent functions.

Exer. 69–72: Use the figure to approximate the following to one decimal place.



69 (a) $\sin 4$ (b) $\sin (-1.2)$

(c) All numbers t between 0 and 2π such that $\sin t = 0.5$

70 (a) $\sin 2$ (b) $\sin (-2.3)$

(c) All numbers t between 0 and 2π such that $\sin t = -0.2$

71 (a) $\cos 4$ (b) $\cos (-1.2)$

(c) All numbers t between 0 and 2π such that $\cos t = -0.6$

72 (a) $\cos 2$ (b) $\cos (-2.3)$

(c) All numbers t between 0 and 2π such that $\cos t = 0.2$

73 **Temperature-humidity relationship** On March 17, 1981, in Tucson, Arizona, the temperature in degrees Fahrenheit could be described by the equation

$$T(t) = -12 \cos \left(\frac{\pi}{12} t\right) + 60,$$

while the relative humidity in percent could be expressed by

$$H(t) = 20 \cos \left(\frac{\pi}{12} t\right) + 60,$$

where t is in hours and $t = 0$ corresponds to 6 A.M.

(a) Construct a table that lists the temperature and relative humidity every three hours, beginning at midnight.

(b) Determine the times when the maximums and minimums occurred for T and H.

(c) Discuss the relationship between the temperature and relative humidity on this day.

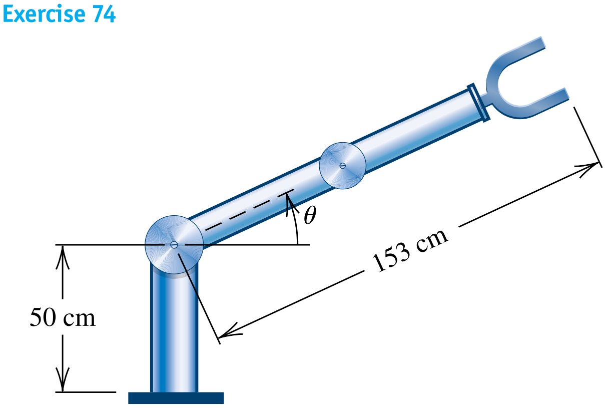
74 **Robotic arm movement** Trigonometric functions are used extensively in the design of industrial robots. Suppose that a robot's shoulder joint is motorized so that the angle θ increases at a constant rate of $\pi/12$ radian per second from an initial angle of $\theta = 0$. Assume that the elbow joint is always kept straight and that the arm has a constant length of 153 centimeters, as shown in the figure.

(a) Assume that $h = 50$ cm when $\theta = 0$. Construct a table that lists the angle θ and the height h of the robotic hand every second while $0 \leq \theta \leq \pi/2$.

(b) Determine whether or not a constant increase in the angle θ produces a constant increase in the height of the hand.

(c) Find the total distance that the hand moves.

Exercise 74



6.4

Values of the Trigonometric Functions

In previous sections we calculated special values of the trigonometric functions by using the definition of the trigonometric functions in terms of either an angle or a unit circle. In practice we most often use a calculator to approximate function values.

We will next show how the value of any trigonometric function at an angle of θ degrees or at a real number t can be found from its value in the θ-interval $(0°, 90°)$ or the t-interval $(0, \pi/2)$, respectively. This technique is sometimes necessary when a calculator is used to find all angles or real numbers that correspond to a given function value.

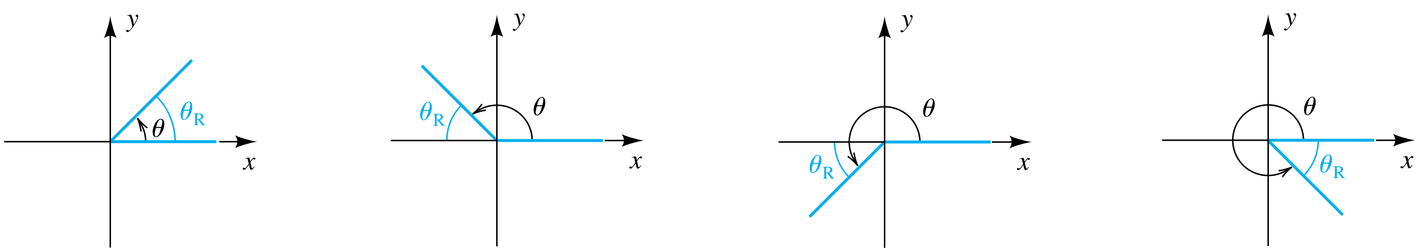
We shall make use of the following concept.

Definition of Reference Angle	Let θ be a nonquadrantal angle in standard position. The **reference angle** for θ is the acute angle θ_R that the terminal side of θ makes with the x-axis.

Figure 1 illustrates the reference angle θ_R for a nonquadrantal angle θ, with $0° < \theta < 360°$ or $0 < \theta < 2\pi$, in each of the four quadrants.

Figure 1 Reference angles

(a) Quadrant I **(b)** Quadrant II **(c)** Quadrant III **(d)** Quadrant IV



(a) $\theta_R = \theta$

(b) $\theta_R = 180° - \theta = \pi - \theta$

(c) $\theta_R = \theta - 180° = \theta - \pi$

(d) $\theta_R = 360° - \theta = 2\pi - \theta$

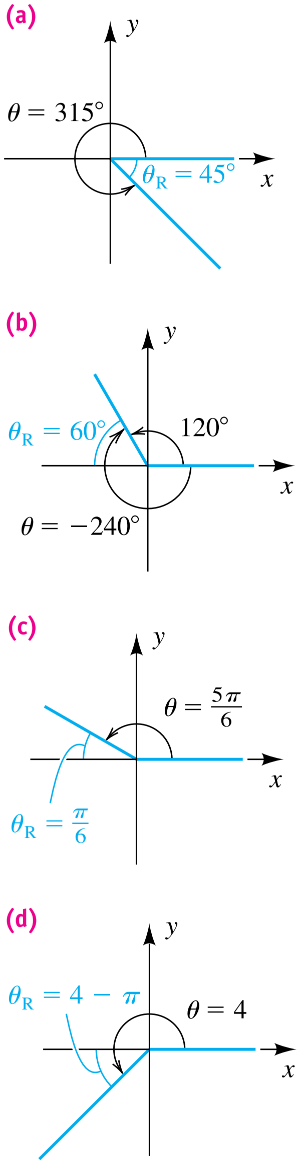
The formulas below the axes in Figure 1 may be used to find the degree or radian measure of θ_R when θ is in degrees or radians, respectively. *For a nonquadrantal angle greater than* $360°$ *or less than* $0°$, *first find the coterminal angle* θ *with* $0° < \theta < 360°$ *or* $0 < \theta < 2\pi$, and then use the formulas in Figure 1.

EXAMPLE 1 Finding reference angles

Find the reference angle θ_R for θ, and sketch θ and θ_R in standard position on the same coordinate plane.

(a) $\theta = 315°$ **(b)** $\theta = -240°$ **(c)** $\theta = \dfrac{5\pi}{6}$ **(d)** $\theta = 4$

Figure 2



SOLUTION

(a) The angle $\theta = 315°$ is in quadrant IV, and hence, as in Figure 1(d),

$$\theta_R = 360° - 315° = 45°.$$

The angles θ and θ_R are sketched in Figure 2(a).

(b) The angle between 0° and 360° that is coterminal with $\theta = -240°$ is

$$-240° + 360° = 120°,$$

which is in quadrant II. Using the formula in Figure 1(b) gives

$$\theta_R = 180° - 120° = 60°.$$

The angles θ and θ_R are sketched in Figure 2(b).

(c) Since the angle $\theta = 5\pi/6$ is in quadrant II, we have

$$\theta_R = \pi - \frac{5\pi}{6} = \frac{\pi}{6},$$

as shown in Figure 2(c).

(d) Since $\pi < 4 < 3\pi/2$, the angle $\theta = 4$ is in quadrant III. Using the formula in Figure 1(c), we obtain

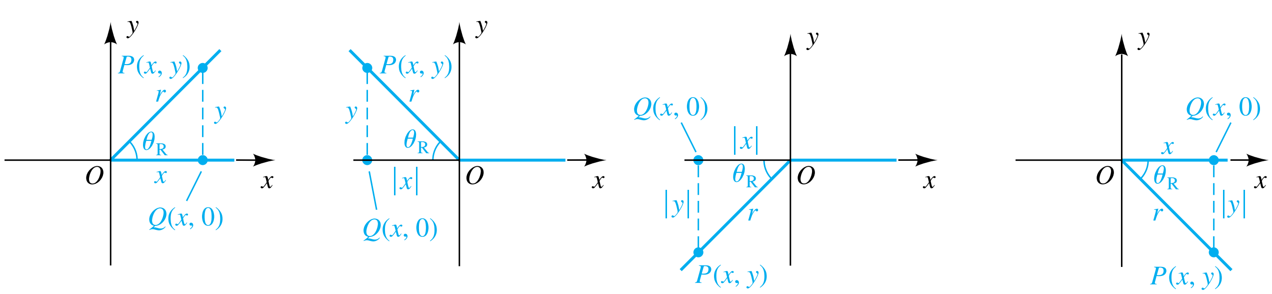
$$\theta_R = 4 - \pi.$$

The angles are sketched in Figure 2(d).

We shall next show how reference angles can be used to find values of the trigonometric functions.

If θ is a nonquadrantal angle with reference angle θ_R, then we have $0° < \theta_R < 90°$ or $0 < \theta_R < \pi/2$. Let $P(x, y)$ be a point on the terminal side of θ, and consider the point $Q(x, 0)$ on the x-axis. Figure 3 illustrates a

Figure 3



typical situation for θ in each quadrant. In each case, the lengths of the sides of triangle OQP are

$$d(O, Q) = |x|, \quad d(Q, P) = |y|, \quad \text{and} \quad d(O, P) = \sqrt{x^2 + y^2} = r.$$

We may apply the definition of the trigonometric functions of any angle and also use triangle OQP to obtain the following formulas:

$$|\sin \theta| = \left|\frac{y}{r}\right| = \frac{|y|}{|r|} = \frac{|y|}{r} = \sin \theta_{\mathrm{R}}$$

$$|\cos \theta| = \left|\frac{x}{r}\right| = \frac{|x|}{|r|} = \frac{|x|}{r} = \cos \theta_{\mathrm{R}}$$

$$|\tan \theta| = \left|\frac{y}{x}\right| = \frac{|y|}{|x|} = \tan \theta_{\mathrm{R}}$$

These formulas lead to the next theorem. If θ is a quadrantal angle, the definition of the trigonometric functions of any angle should be used to find values.

Theorem on Reference Angles

If θ is a nonquadrantal angle in standard position, then to find the value of a trigonometric function at θ, find its value for the reference angle θ_{R} and prefix the appropriate sign.

The "appropriate sign" referred to in the theorem can be determined from the table of signs of the trigonometric functions given on page 371.

EXAMPLE 2 **Using reference angles**

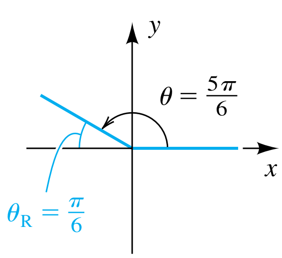
Use reference angles to find the exact values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ if

(a) $\theta = \dfrac{5\pi}{6}$ **(b)** $\theta = 315°$

SOLUTION

(a) The angle $\theta = 5\pi/6$ and its reference angle $\theta_{\mathrm{R}} = \pi/6$ are sketched in Figure 4. Since θ is in quadrant II, $\sin \theta$ is positive and both $\cos \theta$ and $\tan \theta$ are negative. Hence, by the theorem on reference angles and known results about special angles, we obtain the following values:

Figure 4



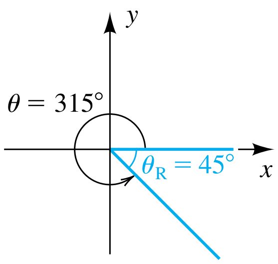
$$\sin \frac{5\pi}{6} = +\sin \frac{\pi}{6} = \frac{1}{2}$$

$$\cos \frac{5\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\tan \frac{5\pi}{6} = -\tan \frac{\pi}{6} = -\frac{\sqrt{3}}{3}$$

(continued)

Figure 5



(b) The angle $\theta = 315°$ and its reference angle $\theta_R = 45°$ are sketched in Figure 5. Since θ is in quadrant IV, $\sin\theta < 0$, $\cos\theta > 0$, and $\tan\theta < 0$. Hence, by the theorem on reference angles, we obtain

$$\sin 315° = -\sin 45° = -\frac{\sqrt{2}}{2}$$

$$\cos 315° = +\cos 45° = \frac{\sqrt{2}}{2}$$

$$\tan 315° = -\tan 45° = -1.$$

If we use a calculator to approximate function values, reference angles are usually unnecessary. As an illustration, to find $\sin 210°$, we place the calculator in degree mode and obtain $\sin 210° = -0.5$, which is the exact value. Using the same procedure for 240°, we obtain a decimal representation:

$$\sin 240° \approx -0.8660$$

A calculator should not be used to find the *exact* value of $\sin 240°$. In this case, we find the reference angle 60° of 240° and use the theorem on reference angles, together with known results about special angles, to obtain

$$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2}.$$

Let us next consider the problem of solving an equation of the following type:

Problem: If θ is an acute angle and $\sin\theta = 0.6635$, approximate θ.

Most calculators have a key labeled $\boxed{\text{SIN}^{-1}}$ that can be used to help solve the equation. With some calculators, it may be necessary to use another key or a keystroke sequence such as $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ (refer to the user manual for your calculator). We shall use the following notation when finding θ, where $0 \le k \le 1$:

$$\text{if} \quad \sin\theta = k, \quad \text{then} \quad \theta = \sin^{-1} k$$

This notation is similar to that used for the inverse function f^{-1} of a function f in Section 5.1, where we saw that under certain conditions,

$$\text{if} \quad f(x) = y, \quad \text{then} \quad x = f^{-1}(y).$$

For the problem $\sin\theta = 0.6635$, f is the sine function, $x = \theta$, and $y = 0.6635$. The notation $\sin^{-1}$ is based on the *inverse trigonometric functions* discussed in Section 7.6. At this stage of our work, *we shall regard* $\sin^{-1}$ *simply as an entry made on a calculator using a* $\boxed{\text{SIN}^{-1}}$ *key.* Thus, for the stated problem, we obtain

$$\theta = \sin^{-1}(0.6635) \approx 41.57° \approx 0.7255.$$

As indicated, when finding an angle, we will usually round off degree measure to the nearest 0.01° and radian measure to four decimal places.

Similarly, given $\cos\theta = k$ or $\tan\theta = k$, where θ is acute, we write

$$\theta = \cos^{-1} k \qquad \text{or} \qquad \theta = \tan^{-1} k$$

to indicate the use of a COS^{-1} or TAN^{-1} key on a calculator.

Given $\csc\theta$, $\sec\theta$, or $\cot\theta$, we use a reciprocal relationship to find θ, as indicated in the following illustration.

ILLUSTRATION **Finding Acute Angle Solutions of Equations with a Calculator**

Equation	Calculator solution (degree and radian)
$\sin\theta = 0.5$	$\theta = \sin^{-1}(0.5) = 30° \approx 0.5236$
$\cos\theta = 0.5$	$\theta = \cos^{-1}(0.5) = 60° \approx 1.0472$
$\tan\theta = 0.5$	$\theta = \tan^{-1}(0.5) \approx 26.57° \approx 0.4636$
$\csc\theta = 2$	$\theta = \sin^{-1}\left(\frac{1}{2}\right) = 30° \approx 0.5236$
$\sec\theta = 2$	$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60° \approx 1.0472$
$\cot\theta = 2$	$\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.57° \approx 0.4636$

The same technique may be employed if θ is *any* angle or real number. Thus, using the SIN^{-1} key, we obtain, in degree or radian mode,

$$\theta = \sin^{-1}(0.6635) \approx 41.57° \approx 0.7255,$$

which is the reference angle for θ. If $\sin\theta$ is *negative,* then a calculator gives us the *negative* of the reference angle. For example,

$$\sin^{-1}(-0.6635) \approx -41.57° \approx -0.7255.$$

Similarly, given $\cos\theta$ or $\tan\theta$, we find θ with a calculator by using COS^{-1} or TAN^{-1}, respectively. The interval containing θ is listed in the next chart. It is important to note that if $\cos\theta$ is negative, then θ is *not* the negative of the reference angle, but instead is in the interval $\pi/2 < \theta \le \pi$, or $90° < \theta \le 180°$. The reasons for using these intervals are explained in Section 7.6. We may use reciprocal relationships to solve similar equations involving $\csc\theta$, $\sec\theta$, and $\cot\theta$.

Equation	Values of k	Calculator solution	Interval containing θ if a calculator is used
$\sin\theta = k$	$-1 \le k \le 1$	$\theta = \sin^{-1} k$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, or $-90° \le \theta \le 90°$
$\cos\theta = k$	$-1 \le k \le 1$	$\theta = \cos^{-1} k$	$0 \le \theta \le \pi$, or $0° \le \theta \le 180°$
$\tan\theta = k$	any k	$\theta = \tan^{-1} k$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, or $-90° < \theta < 90°$

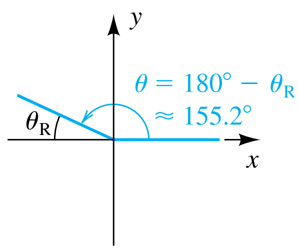
The following illustration contains some specific examples for both degree and radian modes.

ILLUSTRATION **Finding Angles with a Calculator**

Equation	Calculator solution (degree and radian)
■ $\sin\theta = -0.5$	$\theta = \sin^{-1}(-0.5) = -30° \approx -0.5236$
■ $\cos\theta = -0.5$	$\theta = \cos^{-1}(-0.5) = 120° \approx 2.0944$
■ $\tan\theta = -0.5$	$\theta = \tan^{-1}(-0.5) \approx -26.57° \approx -0.4636$

When using a calculator to find θ, be sure to keep the restrictions on θ in mind. If other values are desired, then reference angles or other methods may be employed, as illustrated in the next examples.

Figure 6



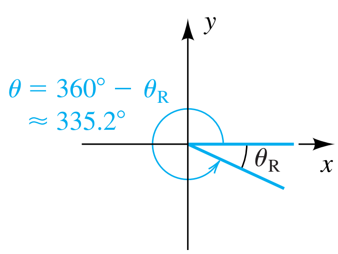
EXAMPLE 3 **Approximating an angle with a calculator**

If $\tan\theta = -0.4623$ and $0° \le \theta < 360°$, find θ to the nearest 0.1°.

SOLUTION As pointed out in the preceding discussion, if we use a calculator (in degree mode) to find θ when $\tan\theta$ is negative, then the degree measure will be in the interval $(-90°, 0°)$. In particular, we obtain the following:

$$\theta = \tan^{-1}(-0.4623) \approx -24.8°$$

Figure 7



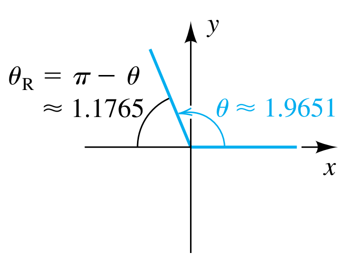
Since we wish to find values of θ between 0° and 360°, we use the (approximate) reference angle $\theta_R \approx 24.8°$. There are two possible values of θ such that $\tan\theta$ is negative—one in quadrant II, the other in quadrant IV. If θ is in quadrant II and $0° \le \theta < 360°$, we have the situation shown in Figure 6, and

$$\theta = 180° - \theta_R \approx 180° - 24.8° = 155.2°.$$

If θ is in quadrant IV and $0° \le \theta < 360°$, then, as in Figure 7,

$$\theta = 360° - \theta_R \approx 360° - 24.8° = 335.2°.$$

Figure 8



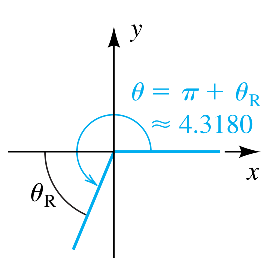
EXAMPLE 4 **Approximating an angle with a calculator**

If $\cos\theta = -0.3842$ and $0 \le \theta < 2\pi$, find θ to the nearest 0.0001 radian.

SOLUTION If we use a calculator (in radian mode) to find θ when $\cos\theta$ is negative, then the radian measure will be in the interval $[0, \pi]$. In particular, we obtain the following (shown in Figure 8):

$$\theta = \cos^{-1}(-0.3842) \approx 1.965\,137\,489$$

Figure 9



Since we wish to find values of θ between 0 and 2π, we use the (approximate) reference angle

$$\theta_R = \pi - \theta \approx 1.176\,455\,165.$$

There are two possible values of θ such that $\cos\theta$ is negative—the one we found in quadrant II and the other in quadrant III. If θ is in quadrant III, then

$$\theta = \pi + \theta_R \approx 4.318\,047\,819,$$

as shown in Figure 9.

6.4 Exercises

Exer. 1–6: Find the reference angle θ_R if θ has the given measure.

1 (a) $240°$ (b) $340°$ (c) $-202°$ (d) $-660°$

2 (a) $165°$ (b) $275°$ (c) $-110°$ (d) $400°$

3 (a) $3\pi/4$ (b) $4\pi/3$ (c) $-\pi/6$ (d) $9\pi/4$

4 (a) $7\pi/4$ (b) $2\pi/3$ (c) $-3\pi/4$ (d) $-23\pi/6$

5 (a) 3 (b) -2 (c) 5.5 (d) 100

6 (a) 6 (b) -4 (c) 4.5 (d) 80

Exer. 7–18: Find the exact value.

7 (a) $\sin(2\pi/3)$ (b) $\sin(-5\pi/4)$

8 (a) $\sin 210°$ (b) $\sin(-315°)$

9 (a) $\cos 150°$ (b) $\cos(-60°)$

10 (a) $\cos(5\pi/4)$ (b) $\cos(-11\pi/6)$

11 (a) $\tan(5\pi/6)$ (b) $\tan(-\pi/3)$

12 (a) $\tan 330°$ (b) $\tan(-225°)$

13 (a) $\cot 120°$ (b) $\cot(-150°)$

14 (a) $\cot(3\pi/4)$ (b) $\cot(-2\pi/3)$

15 (a) $\sec(2\pi/3)$ (b) $\sec(-\pi/6)$

16 (a) $\sec 135°$ (b) $\sec(-210°)$

17 (a) $\csc 240°$ (b) $\csc(-330°)$

18 (a) $\csc(3\pi/4)$ (b) $\csc(-2\pi/3)$

Exer. 19–24: Approximate to three decimal places.

19 (a) $\sin 73°20'$ (b) $\cos 0.68$

20 (a) $\cos 38°30'$ (b) $\sin 1.48$

21 (a) $\tan 21°10'$ (b) $\cot 1.13$

22 (a) $\cot 9°10'$ (b) $\tan 0.75$

23 (a) $\sec 67°50'$ (b) $\csc 0.32$

24 (a) $\csc 43°40'$ (b) $\sec 0.26$

Exer. 25–32: Approximate the acute angle θ to the nearest (a) 0.01° and (b) 1′.

25 $\cos\theta = 0.8620$

26 $\sin\theta = 0.6612$

27 $\tan\theta = 3.7$

28 $\cos\theta = 0.8$

29 $\sin\theta = 0.4217$

30 $\tan\theta = 4.91$

31 $\sec\theta = 4.246$

32 $\csc\theta = 11$

Exer. 33–34: Approximate to four decimal places.

33 (a) $\sin 98°10'$ (b) $\cos 623.7°$ (c) $\tan 3$
(d) $\cot 231°40'$ (e) $\sec 1175.1°$ (f) $\csc 0.82$

34 (a) $\sin 496.4°$ (b) $\cos 0.65$ (c) $\tan 105°40'$
(d) $\cot 1030.2°$ (e) $\sec 1.46$ (f) $\csc 320°50'$

Exer. 35–36: Approximate, to the nearest 0.1°, all angles θ in the interval $[0°, 360°)$ that satisfy the equation.

35 (a) $\sin\theta = -0.5640$ (b) $\cos\theta = 0.7490$
(c) $\tan\theta = 2.798$ (d) $\cot\theta = -0.9601$
(e) $\sec\theta = -1.116$ (f) $\csc\theta = 1.485$

36 (a) $\sin\theta = 0.8225$ (b) $\cos\theta = -0.6604$
(c) $\tan\theta = -1.5214$ (d) $\cot\theta = 1.3752$
(e) $\sec\theta = 1.4291$ (f) $\csc\theta = -2.3179$

Exer. 37–38: Approximate, to the nearest 0.01 radian, all angles θ in the interval $[0, 2\pi)$ that satisfy the equation.

37 (a) $\sin\theta = 0.4195$ (b) $\cos\theta = -0.1207$
(c) $\tan\theta = -3.2504$ (d) $\cot\theta = 2.6815$
(e) $\sec\theta = 1.7452$ (f) $\csc\theta = -4.8521$

38 (a) $\sin\theta = -0.0135$ (b) $\cos\theta = 0.9235$
(c) $\tan\theta = 0.42$ (d) $\cot\theta = -2.731$
(e) $\sec\theta = -3.51$ (f) $\csc\theta = 1.258$

39 Thickness of the ozone layer The thickness of the ozone layer can be estimated using the formula

$$\ln I_0 - \ln I = kx \sec \theta,$$

where I_0 is the intensity of a particular wavelength of light from the sun before it reaches the atmosphere, I is the intensity of the same wavelength after passing through a layer of ozone x centimeters thick, k is the absorption constant of ozone for that wavelength, and θ is the acute angle that the sunlight makes with the vertical. Suppose that for a wavelength of 3055×10^{-8} centimeter with $k \approx 1.88$, I_0/I is measured as 1.72 and $\theta = 12°$. Approximate the thickness of the ozone layer to the nearest 0.01 centimeter.

40 Ozone calculations Refer to Exercise 39. If the ozone layer is estimated to be 0.31 centimeter thick and, for a wavelength of 3055×10^{-8} centimeter, I_0/I is measured as 2.05, approximate the angle the sun made with the vertical at the time of the measurement.

41 Solar radiation The amount of sunshine illuminating a wall of a building can greatly affect the energy efficiency of the building. The solar radiation striking a vertical wall that faces east is given by the formula

$$R = R_0 \cos \theta \sin \phi,$$

where R_0 is the maximum solar radiation possible, θ is the angle that the sun makes with the horizontal, and ϕ is the direction of the sun in the sky, with $\phi = 90°$ when the sun is in the east and $\phi = 0°$ when the sun is in the south.

(a) When does the maximum solar radiation R_0 strike the wall?

(b) What percentage of R_0 is striking the wall when θ is equal to 60° and the sun is in the southeast?

42 Meteorological calculations In the mid-latitudes it is sometimes possible to estimate the distance between consecutive regions of low pressure. If ϕ is the latitude (in degrees), R is Earth's radius (in kilometers), and v is the horizontal wind velocity (in km/hr), then the distance d (in kilometers) from one low pressure area to the next can be estimated using the formula

$$d = 2\pi\left(\frac{vR}{0.52 \cos \phi}\right)^{1/3}.$$

(a) At a latitude of 48°, Earth's radius is approximately 6369 kilometers. Approximate d if the wind speed is 45 km/hr.

(b) If v and R are constant, how does d vary as the latitude increases?

43 Robot's arm Points on the terminal sides of angles play an important part in the design of arms for robots. Suppose a robot has a straight arm 18 inches long that can rotate about the origin in a coordinate plane. If the robot's hand is located at (18, 0) and then rotates through an angle of 60°, what is the new location of the hand?

44 Robot's arm Suppose the robot's arm in Exercise 43 can change its length in addition to rotating about the origin. If the hand is initially at (12, 12), approximately how many degrees should the arm be rotated and how much should its length be changed to move the hand to (−16, 10)?

6.5 Trigonometric Graphs

In this section we consider graphs of the equations

$$y = a \sin (bx + c) \qquad \text{and} \qquad y = a \cos (bx + c)$$

for real numbers a, b, and c. Our goal is to sketch such graphs without plotting many points. To do so we shall use facts about the graphs of the sine and cosine functions discussed in Section 6.3.

Let us begin by considering the special case $c = 0$ and $b = 1$—that is,

$$y = a \sin x \qquad \text{and} \qquad y = a \cos x.$$

We can find y-coordinates of points on the graphs by multiplying y-coordinates of points on the graphs of $y = \sin x$ and $y = \cos x$ by a. To illustrate, if $y = 2 \sin x$, we multiply the y-coordinate of each point on the graph of

$y = \sin x$ by 2. This gives us Figure 1, where for comparison we also show the graph of $y = \sin x$. The procedure is the same as that for vertically stretching the graph of a function, discussed in Section 3.5.

As another illustration, if $y = \frac{1}{2} \sin x$, we multiply y-coordinates of points on the graph of $y = \sin x$ by $\frac{1}{2}$. This multiplication vertically compresses the graph of $y = \sin x$ by a factor of 2, as illustrated in Figure 2.

Figure 1

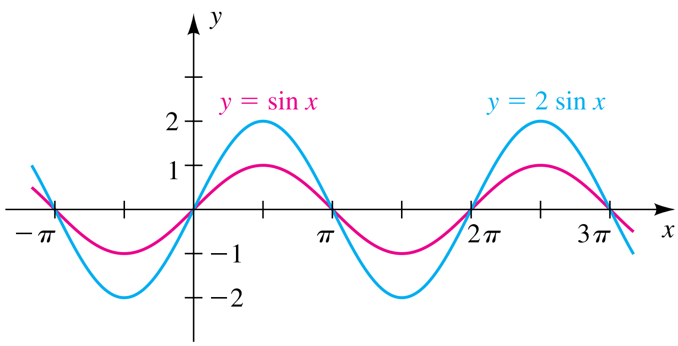
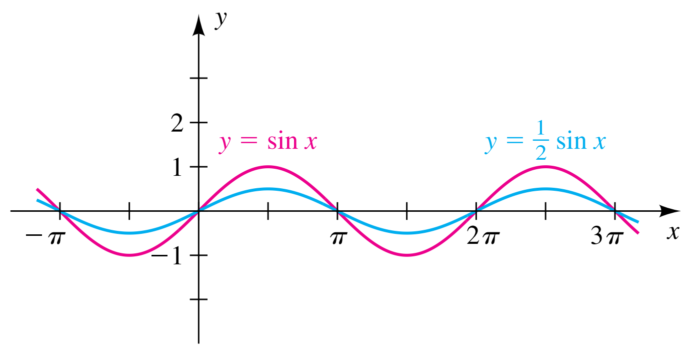


Figure 2



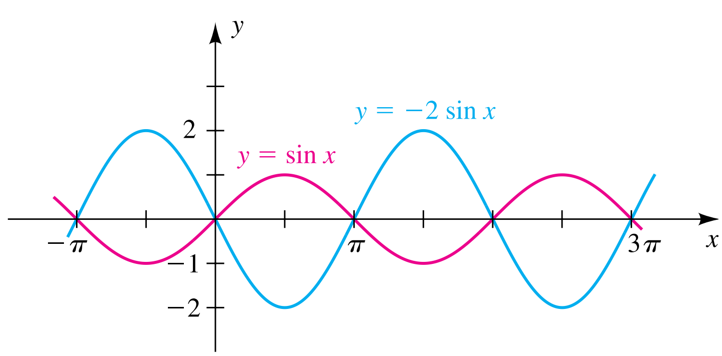
The following example illustrates a graph of $y = a \sin x$ with a negative.

EXAMPLE 1 **Sketching the graph of an equation involving sin x**

Sketch the graph of the equation $y = -2 \sin x$.

SOLUTION The graph of $y = -2 \sin x$ sketched in Figure 3 can be obtained by first sketching the graph of $y = \sin x$ (shown in the figure) and then multiplying y-coordinates by -2. An alternative method is to reflect the graph of $y = 2 \sin x$ (see Figure 1) through the x-axis.

Figure 3



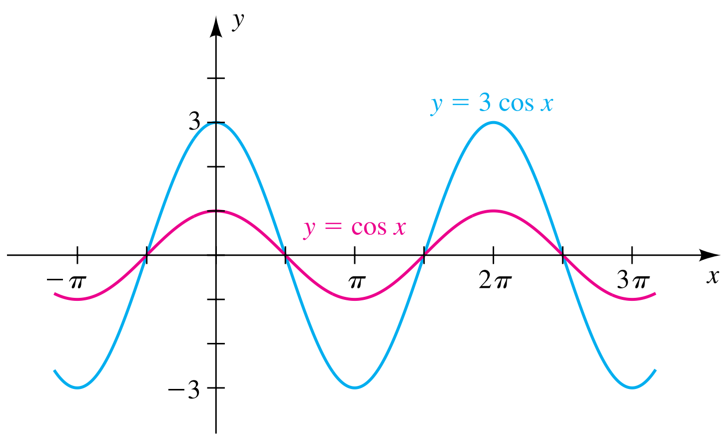
For any $a \neq 0$, the graph of $y = a \sin x$ has the general appearance of one of the graphs illustrated in Figures 1, 2, and 3. The amount of stretching of the graph of $y = \sin x$ and whether the graph is reflected are determined by the absolute value of a and the sign of a, respectively. The largest y-coordinate $|a|$ is the **amplitude of the graph** or, equivalently, the **amplitude of the function** ***f*** given by $f(x) = a \sin x$. In Figures 1 and 3 the amplitude is 2. In Figure 2 the amplitude is $\frac{1}{2}$. Similar remarks and techniques apply if $y = a \cos x$.

EXAMPLE 2 **Sketching the graph of an equation involving $\cos x$**

Find the amplitude and sketch the graph of $y = 3 \cos x$.

SOLUTION By the preceding discussion, the amplitude is 3. As indicated in Figure 4, we first sketch the graph of $y = \cos x$ and then multiply y-coordinates by 3.

Figure 4



Let us next consider $y = a \sin bx$ and $y = a \cos bx$ for nonzero real numbers a and b. As before, the amplitude is $|a|$. If $b > 0$, then exactly one cycle occurs as bx increases from 0 to 2π or, equivalently, as x increases from 0 to $2\pi/b$. If $b < 0$, then $-b > 0$ and one cycle occurs as x increases from 0 to $2\pi/(-b)$. Thus, the period of the function f given by $f(x) = a \sin bx$ or $f(x) = a \cos bx$ is $2\pi/|b|$. For convenience, we shall also refer to $2\pi/|b|$ as the period of the *graph of f*. The next theorem summarizes our discussion.

Theorem on Amplitudes and Periods	If $y = a \sin bx$ or $y = a \cos bx$ for nonzero real numbers a and b, then the graph has amplitude $|a|$ and period $\dfrac{2\pi}{|b|}$.

We can also relate the role of b to the discussion of horizontally compressing and stretching a graph in Section 3.5. If $|b| > 1$, the graph of $y = \sin bx$ or $y = \cos bx$ can be considered to be compressed horizontally by a factor b. If $0 < |b| < 1$, the graphs are stretched horizontally by a factor $1/b$. This concept is illustrated in the next two examples.

EXAMPLE 3 Finding an amplitude and a period

Find the amplitude and the period and sketch the graph of $y = 3 \sin 2x$.

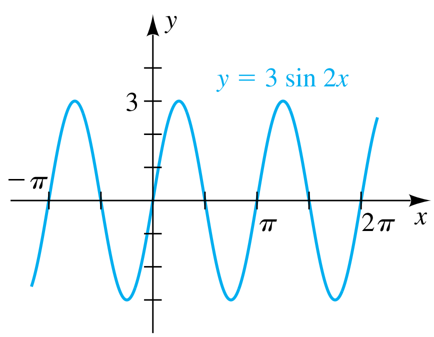
SOLUTION Using the theorem on amplitudes and periods with $a = 3$ and $b = 2$, we obtain the following:

$$\text{amplitude: } |a| = |3| = 3$$

$$\text{period: } \frac{2\pi}{|b|} = \frac{2\pi}{|2|} = \frac{2\pi}{2} = \pi$$

Thus, there is exactly one sine wave of amplitude 3 on the x-interval $[0, \pi]$. Sketching this wave and then extending the graph to the right and left gives us Figure 5.

Figure 5



EXAMPLE 4 Finding an amplitude and a period

Find the amplitude and the period and sketch the graph of $y = 2 \sin \frac{1}{2}x$.

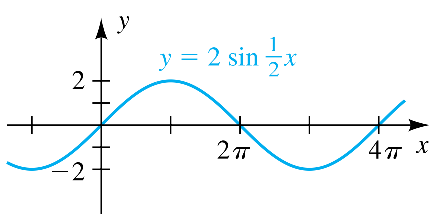
SOLUTION Using the theorem on amplitudes and periods with $a = 2$ and $b = \frac{1}{2}$, we obtain the following:

$$\text{amplitude: } |a| = |2| = 2$$

$$\text{period: } \frac{2\pi}{|b|} = \frac{2\pi}{|\frac{1}{2}|} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

Thus, there is one sine wave of amplitude 2 on the interval $[0, 4\pi]$. Sketching this wave and extending it left and right gives us the graph in Figure 6.

Figure 6



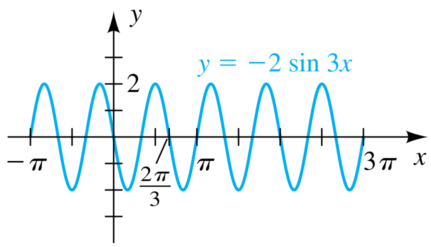
If $y = a \sin bx$ and if b is a large positive number, then the period $2\pi/b$ is small and the sine waves are close together, with b sine waves on the interval $[0, 2\pi]$. For example, in Figure 5, $b = 2$ and we have two sine waves on $[0, 2\pi]$. If b is a small positive number, then the period $2\pi/b$ is large and the waves are far apart. To illustrate, if $y = \sin \frac{1}{10}x$, then one-tenth of a sine wave occurs on $[0, 2\pi]$ and an interval 20π units long is required for one complete cycle. (See also Figure 6—for $y = 2 \sin \frac{1}{2}x$, one-half of a sine wave occurs on $[0, 2\pi]$.)

If $b < 0$, we can use the fact that $\sin(-x) = -\sin x$ to obtain the graph of $y = a \sin bx$. To illustrate, the graph of $y = \sin(-2x)$ is the same as the graph of $y = -\sin 2x$.

EXAMPLE 5 **Finding an amplitude and a period**

Find the amplitude and the period and sketch the graph of the equation $y = 2 \sin(-3x)$.

Figure 7



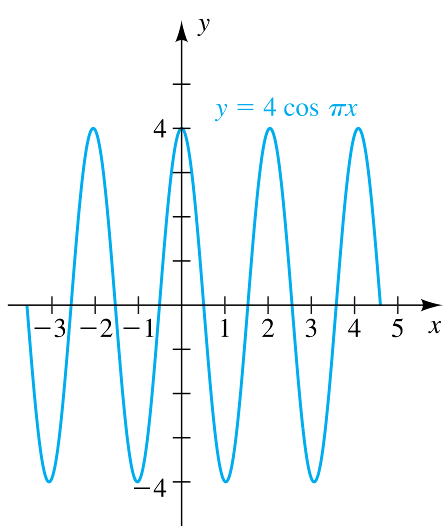
SOLUTION Since the sine function is odd, $\sin(-3x) = -\sin 3x$, and we may write the equation as $y = -2 \sin 3x$. The amplitude is $|-2| = 2$, and the period is $2\pi/3$. Thus, there is one cycle on an interval of length $2\pi/3$. The negative sign indicates a reflection through the x-axis. If we consider the interval $[0, 2\pi/3]$ and sketch a sine wave of amplitude 2 (reflected through the x-axis), the shape of the graph is apparent. The part of the graph in the interval $[0, 2\pi/3]$ is repeated periodically, as illustrated in Figure 7.

EXAMPLE 6 **Finding an amplitude and a period**

Find the amplitude and the period and sketch the graph of $y = 4 \cos \pi x$.

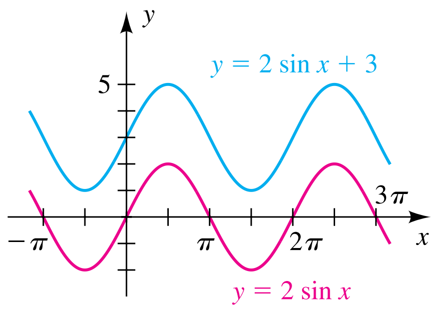
SOLUTION The amplitude is $|4| = 4$, and the period is $2\pi/\pi = 2$. Thus, there is exactly one cosine wave of amplitude 4 on the interval $[0, 2]$. Since the period does not contain the number π, it makes sense to use integer ticks on the x-axis. Sketching this wave and extending it left and right gives us the graph in Figure 8.

Figure 8



As discussed in Section 3.5, if f is a function and c is a positive real number, then the graph of $y = f(x) + c$ can be obtained by shifting the graph of $y = f(x)$ vertically upward a distance c. For the graph of $y = f(x) - c$, we shift the graph of $y = f(x)$ vertically downward a distance of c. In the next example we use this technique for a trigonometric graph.

Figure 9



EXAMPLE 7 **Vertically shifting a trigonometric graph**

Sketch the graph of $y = 2 \sin x + 3$.

SOLUTION It is important to note that $y \neq 2 \sin (x + 3)$. The graph of $y = 2 \sin x$ is sketched in red in Figure 9. If we shift this graph vertically upward a distance 3, we obtain the graph of $y = 2 \sin x + 3$.

Let us next consider the graph of

$$y = a \sin (bx + c).$$

As before, the amplitude is $|a|$, and the period is $2\pi/|b|$. One cycle occurs if $bx + c$ increases from 0 to 2π. Hence, we can find an interval containing exactly one sine wave by solving the following inequality for x:

$$\begin{aligned} 0 \leq bx + c &\leq 2\pi \\ -c \leq bx \quad &\leq 2\pi - c && \text{subtract } c \\ -\frac{c}{b} \leq \quad x \quad &\leq \frac{2\pi}{b} - \frac{c}{b} && \text{divide by } b \end{aligned}$$

The number $-c/b$ is the **phase shift** associated with the graph. The graph of $y = a \sin (bx + c)$ may be obtained by shifting the graph of $y = a \sin bx$ to the left if the phase shift is negative or to the right if the phase shift is positive.

Analogous results are true for $y = a \cos (bx + c)$. The next theorem summarizes our discussion.

Theorem on Amplitudes, Periods, and Phase Shifts

If $y = a \sin (bx + c)$ or $y = a \cos (bx + c)$ for nonzero real numbers a and b, then

(1) the amplitude is $|a|$, the period is $\dfrac{2\pi}{|b|}$, and the phase shift is $-\dfrac{c}{b}$;

(2) an interval containing exactly one cycle can be found by solving the inequality

$$0 \leq bx + c \leq 2\pi.$$

We will sometimes write $y = a \sin (bx + c)$ in the equivalent form $y = a \sin \left[b\left(x + \dfrac{c}{b} \right) \right]$.

EXAMPLE 8 **Finding an amplitude, a period, and a phase shift**

Find the amplitude, the period, and the phase shift and sketch the graph of

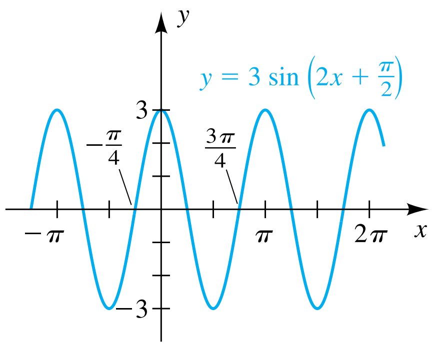
$$y = 3 \sin \left(2x + \frac{\pi}{2} \right).$$

SOLUTION The equation is of the form $y = a \sin(bx + c)$ with $a = 3$, $b = 2$, and $c = \pi/2$. Thus, the amplitude is $|a| = 3$, and the period is $2\pi/|b| = 2\pi/2 = \pi$.

By part (2) of the theorem on amplitudes, periods, and phase shifts, the phase shift and an interval containing one sine wave can be found by solving the following inequality:

$$\begin{aligned} 0 \le 2x + \frac{\pi}{2} &\le 2\pi \\ -\frac{\pi}{2} \le 2x \quad &\le \frac{3\pi}{2} \quad \text{subtract } \frac{\pi}{2} \\ -\frac{\pi}{4} \le \ x \quad &\le \frac{3\pi}{4} \quad \text{divide by 2} \end{aligned}$$

Figure 10



Thus, the phase shift is $-\pi/4$, and one sine wave of amplitude 3 occurs on the interval $[-\pi/4, 3\pi/4]$. Sketching that wave and then repeating it to the right and left gives us the graph in Figure 10.

EXAMPLE 9 Finding an amplitude, a period, and a phase shift

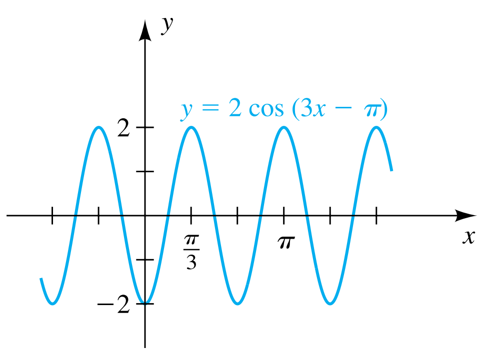
Find the amplitude, the period, and the phase shift and sketch the graph of $y = 2 \cos(3x - \pi)$.

SOLUTION The equation has the form $y = a \cos(bx + c)$ with $a = 2$, $b = 3$, and $c = -\pi$. Thus, the amplitude is $|a| = 2$, and the period is $2\pi/|b| = 2\pi/3$.

By part (2) of the theorem on amplitudes, periods, and phase shifts, the phase shift and an interval containing one cycle can be found by solving the following inequality:

$$\begin{aligned} 0 \le 3x - \pi &\le 2\pi \\ \pi \le 3x \quad &\le 3\pi \quad \text{add } \pi \\ \frac{\pi}{3} \le \ x \quad &\le \pi \quad \text{divide by 3} \end{aligned}$$

Figure 11



Hence, the phase shift is $\pi/3$, and one cosine-type cycle (from maximum to maximum) of amplitude 2 occurs on the interval $[\pi/3, \pi]$. Sketching that part of the graph and then repeating it to the right and left gives us the sketch in Figure 11.

If we solve the inequality

$$-\frac{\pi}{2} \le 3x - \pi \le \frac{3\pi}{2} \qquad \text{instead of} \qquad 0 \le 3x - \pi \le 2\pi,$$

we obtain the interval $\pi/6 \le x \le 5\pi/6$, which gives us a cycle between x-intercepts rather than a cycle between maximums.

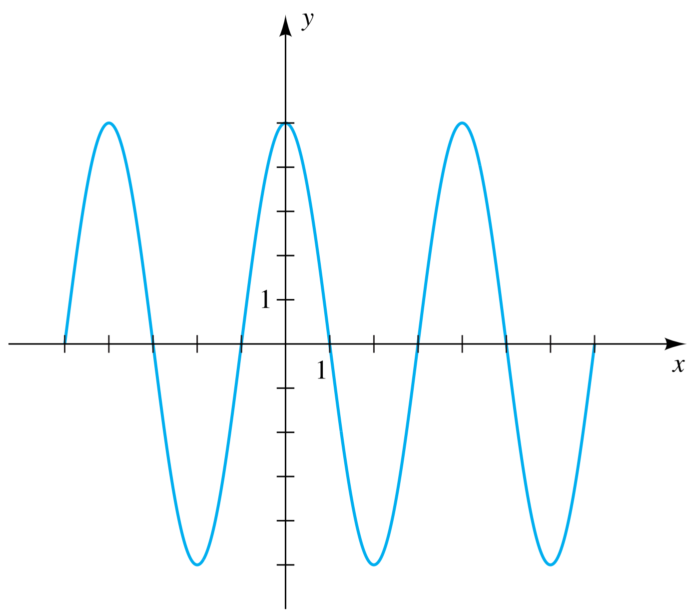
EXAMPLE 10 **Finding an equation for a sine wave**

Express the equation for the sine wave shown in Figure 12 in the form

$$y = a \sin (bx + c)$$

for $a > 0$, $b > 0$, and the least positive real number c.

Figure 12



SOLUTION The largest and smallest y-coordinates of points on the graph are 5 and -5, respectively. Hence, the amplitude is $a = 5$.

Since one sine wave occurs on the interval $[-1, 3]$, the period has value $3 - (-1) = 4$. Hence, by the theorem on amplitudes, periods, and phase shifts (with $b > 0$),

$$\frac{2\pi}{b} = 4 \qquad \text{or, equivalently,} \qquad b = \frac{\pi}{2}.$$

The phase shift is $-c/b = -c/(\pi/2)$. Since c is to be positive, the phase shift must be *negative;* that is, the graph in Figure 12 must be obtained by shifting the graph of $y = 5 \sin [(\pi/2)x]$ to the *left.* Since we want c to be as small as possible, we choose the phase shift -1. Hence,

$$-\frac{c}{\pi/2} = -1 \qquad \text{or, equivalently,} \qquad c = \frac{\pi}{2}.$$

Thus, the desired equation is

$$y = 5 \sin \left(\frac{\pi}{2}x + \frac{\pi}{2}\right).$$

(continued)

There are many other equations for the graph. For example, we could use the phase shifts -5, -9, -13, and so on, but these would not give us the *least* positive value for c. Two other equations for the graph are

$$y = 5\sin\left(\frac{\pi}{2}x - \frac{3\pi}{2}\right) \qquad \text{and} \qquad y = -5\sin\left(\frac{\pi}{2}x + \frac{3\pi}{2}\right).$$

However, neither of these equations satisfies the given criteria for a, b, and c, since in the first, $c < 0$, and in the second, $a < 0$ and c does not have its least positive value.

As an alternative solution, we could write

$$y = a\sin(bx + c) \qquad \text{as} \qquad y = a\sin\left[b\left(x + \frac{c}{b}\right)\right].$$

As before, we find $a = 5$ and $b = \pi/2$. Now since the graph has an x-intercept at $x = -1$, we can consider this graph to be a horizontal shift of the graph of $y = 5\sin[(\pi/2)x]$ to the left by 1 unit—that is, replace x with $x + 1$. Thus, an equation is

$$y = 5\sin\left[\frac{\pi}{2}(x + 1)\right], \qquad \text{or} \qquad y = 5\sin\left(\frac{\pi}{2}x + \frac{\pi}{2}\right).$$

Many phenomena that occur in nature vary in a cyclic or rhythmic manner. It is sometimes possible to represent such behavior by means of trigonometric functions, as illustrated in the next two examples.

EXAMPLE 11 Analyzing the process of breathing

The rhythmic process of breathing consists of alternating periods of inhaling and exhaling. One complete cycle normally takes place every 5 seconds. If $F(t)$ denotes the air flow rate at time t (in liters per second) and if the maximum flow rate is 0.6 liter per second, find a formula of the form $F(t) = a\sin bt$ that fits this information.

SOLUTION If $F(t) = a\sin bt$ for some $b > 0$, then the period of F is $2\pi/b$. In this application the period is 5 seconds, and hence

$$\frac{2\pi}{b} = 5, \qquad \text{or} \qquad b = \frac{2\pi}{5}.$$

Since the maximum flow rate corresponds to the amplitude a of F, we let $a = 0.6$. This gives us the formula

$$F(t) = 0.6\sin\left(\frac{2\pi}{5}t\right).$$

EXAMPLE 12 **Approximating the number of hours of daylight in a day**

The number of hours of daylight $D(t)$ at a particular time of the year can be approximated by

$$D(t) = \frac{K}{2}\sin\left[\frac{2\pi}{365}(t - 79)\right] + 12$$

for t in days and $t = 0$ corresponding to January 1. The constant K determines the total variation in day length and depends on the latitude of the locale.

(a) For Boston, $K \approx 6$. Sketch the graph of D for $0 \le t \le 365$.

(b) When is the day length the longest? the shortest?

SOLUTION

(a) If $K = 6$, then $K/2 = 3$, and we may write $D(t)$ in the form

$$D(t) = f(t) + 12,$$

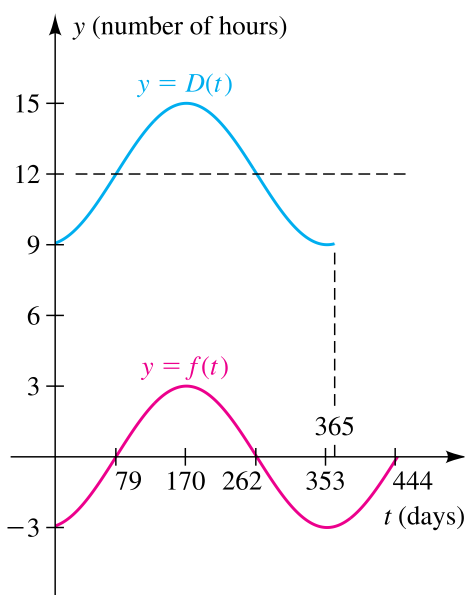
where
$$f(t) = 3\sin\left[\frac{2\pi}{365}(t - 79)\right].$$

We shall sketch the graph of f and then apply a vertical shift through a distance 12.

As in part (2) of the theorem on amplitudes, periods, and phase shifts, we can obtain a t-interval containing exactly one cycle by solving the following inequality:

$$\begin{aligned} 0 \le \frac{2\pi}{365}&(t - 79) \le 2\pi \\ 0 \le\ & t - 79 \le 365 && \text{multiply by } \frac{365}{2\pi} \\ 79 \le\ & t \le 444 && \text{add 79} \end{aligned}$$

Figure 13



Hence, one sine wave occurs on the interval [79, 444]. Dividing this interval into four equal parts, we obtain the following table of values, which indicates the familiar sine wave pattern of amplitude 3.

t	79	170.25	261.5	352.75	444
$f(t)$	0	3	0	−3	0

If $t = 0$,

$$f(0) = 3\sin\left[\frac{2\pi}{365}(-79)\right] \approx 3\sin(-1.36) \approx -2.9.$$

Since the period of f is 365, this implies that $f(365) \approx -2.9$.

The graph of f for the interval [0, 444] is sketched in Figure 13, with different scales on the axes and t rounded off to the nearest day.

(continued)

Applying a vertical shift of 12 units gives us the graph of D for $0 \leq t \leq 365$ shown in Figure 13.

(b) The longest day—that is, the largest value of $D(t)$—occurs 170 days after January 1. Except for leap year, this corresponds to June 20. The shortest day occurs 353 days after January 1, or December 20.

6.5 *Exercises*

1 Find the amplitude and the period and sketch the graph of the equation:

(a) $y = 4 \sin x$ (b) $y = \sin 4x$

(c) $y = \frac{1}{4} \sin x$ (d) $y = \sin \frac{1}{4}x$

(e) $y = 2 \sin \frac{1}{4}x$ (f) $y = \frac{1}{2} \sin 4x$

(g) $y = -4 \sin x$ (h) $y = \sin (-4x)$

2 For equations analogous to those in (a)–(h) of Exercise 1 but involving the cosine, find the amplitude and the period and sketch the graph.

3 Find the amplitude and the period and sketch the graph of the equation:

(a) $y = 3 \cos x$ (b) $y = \cos 3x$

(c) $y = \frac{1}{3} \cos x$ (d) $y = \cos \frac{1}{3}x$

(e) $y = 2 \cos \frac{1}{3}x$ (f) $y = \frac{1}{2} \cos 3x$

(g) $y = -3 \cos x$ (h) $y = \cos (-3x)$

4 For equations analogous to those in (a)–(h) of Exercise 3 but involving the sine, find the amplitude and the period and sketch the graph.

Exer. 5–40: Find the amplitude, the period, and the phase shift and sketch the graph of the equation.

5 $y = \sin \left(x - \dfrac{\pi}{2}\right)$ 6 $y = \sin \left(x + \dfrac{\pi}{4}\right)$

7 $y = 3 \sin \left(x + \dfrac{\pi}{6}\right)$ 8 $y = 2 \sin \left(x - \dfrac{\pi}{3}\right)$

9 $y = \cos \left(x + \dfrac{\pi}{2}\right)$ 10 $y = \cos \left(x - \dfrac{\pi}{3}\right)$

11 $y = 4 \cos \left(x - \dfrac{\pi}{4}\right)$ 12 $y = 3 \cos \left(x + \dfrac{\pi}{6}\right)$

13 $y = \sin (2x - \pi) + 1$ 14 $y = -\sin (3x + \pi) - 1$

15 $y = -\cos (3x + \pi) - 2$ 16 $y = \cos (2x - \pi) + 2$

17 $y = -2 \sin (3x - \pi)$ 18 $y = 3 \cos (3x - \pi)$

19 $y = \sin \left(\dfrac{1}{2}x - \dfrac{\pi}{3}\right)$ 20 $y = \sin \left(\dfrac{1}{2}x + \dfrac{\pi}{4}\right)$

21 $y = 6 \sin \pi x$ 22 $y = 3 \cos \dfrac{\pi}{2}x$

23 $y = 2 \cos \dfrac{\pi}{2}x$ 24 $y = 4 \sin 3\pi x$

25 $y = \dfrac{1}{2} \sin 2\pi x$ 26 $y = \dfrac{1}{2} \cos \dfrac{\pi}{2}x$

27 $y = 5 \sin \left(3x - \dfrac{\pi}{2}\right)$ 28 $y = -4 \cos \left(2x + \dfrac{\pi}{3}\right)$

29 $y = 3 \cos \left(\dfrac{1}{2}x - \dfrac{\pi}{4}\right)$ 30 $y = -2 \sin \left(\dfrac{1}{2}x + \dfrac{\pi}{2}\right)$

31 $y = -5 \cos \left(\dfrac{1}{3}x + \dfrac{\pi}{6}\right)$ 32 $y = 4 \sin \left(\dfrac{1}{3}x - \dfrac{\pi}{3}\right)$

33 $y = 3 \cos (\pi x + 4\pi)$ 34 $y = -2 \sin (2\pi x + \pi)$

35 $y = -\sqrt{2} \sin \left(\dfrac{\pi}{2}x - \dfrac{\pi}{4}\right)$

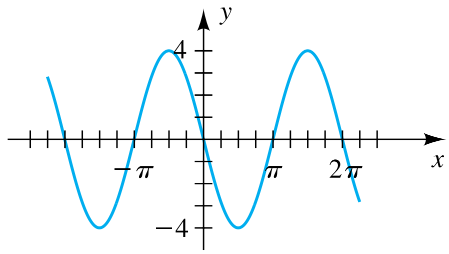
36 $y = \sqrt{3} \cos \left(\dfrac{\pi}{4}x - \dfrac{\pi}{2}\right)$

37 $y = -2 \sin (2x - \pi) + 3$ 38 $y = 3 \cos (x + 3\pi) - 2$

39 $y = 5 \cos (2x + 2\pi) + 2$ 40 $y = -4 \sin (3x - \pi) - 3$

Exer. 41–44: The graph of an equation is shown in the figure. (a) Find the amplitude, period, and phase shift. (b) Write the equation in the form $y = a \sin(bx + c)$ for $a > 0$, $b > 0$, and the least positive real number c.

41



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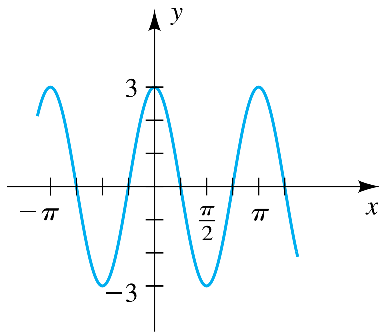

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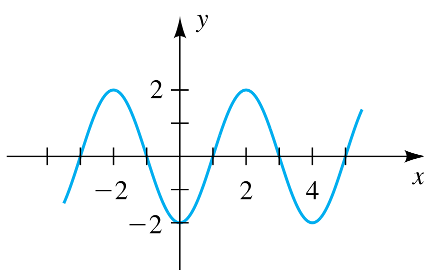

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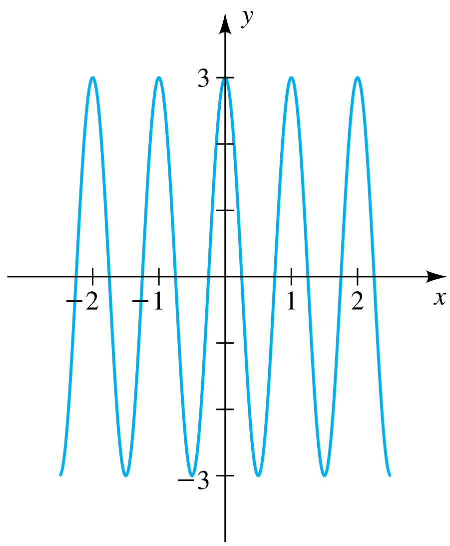
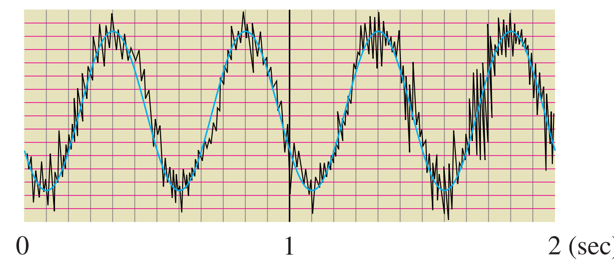


45 **Electroencephalography** Shown in the figure is an electroencephalogram of human brain waves during deep sleep. If we use $W = a \sin(bt + c)$ to represent these waves, what is the value of b?

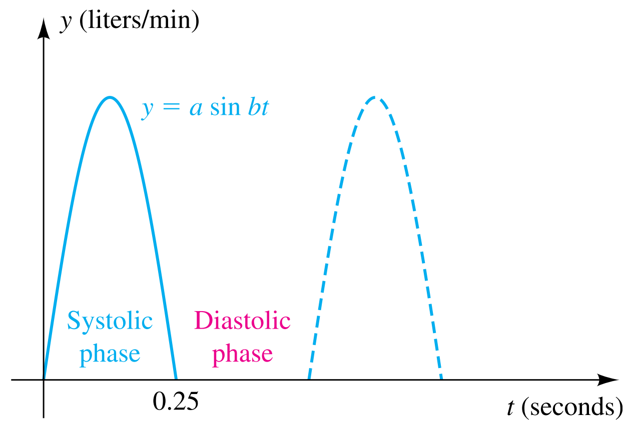
Exercise 45



46 **Intensity of daylight** On a certain spring day with 12 hours of daylight, the light intensity I takes on its largest value of 510 calories/cm^2 at midday. If $t = 0$ corresponds to sunrise, find a formula $I = a \sin bt$ that fits this information.

47 **Heart action** The pumping action of the heart consists of the systolic phase, in which blood rushes from the left ventricle into the aorta, and the diastolic phase, during which the heart muscle relaxes. The function whose graph is shown in the figure is sometimes used to model one complete cycle of this process. For a particular individual, the systolic phase lasts $\frac{1}{4}$ second and has a maximum flow rate of 8 liters per minute. Find a and b.

Exercise 47



48 **Biorhythms** The popular biorhythm theory uses the graphs of three simple sine functions to make predictions about an individual's physical, emotional, and intellectual potential for a particular day. The graphs are given by $y = a \sin bt$

(continued)

for t in days, with $t = 0$ corresponding to birth and $a = 1$ denoting 100% potential.

(a) Find the value of b for the physical cycle, which has a period of 23 days; for the emotional cycle (period 28 days); and for the intellectual cycle (period 33 days).

(b) Evaluate the biorhythm cycles for a person who has just become 21 years of age and is exactly 7670 days old.

49 **Tidal components** The height of the tide at a particular point on shore can be predicted by using seven trigonometric functions (called tidal components) of the form

$$f(t) = a \cos(bt + c).$$

The principal lunar component may be approximated by

$$f(t) = a \cos\left(\frac{\pi}{6}t - \frac{11\pi}{12}\right),$$

where t is in hours and $t = 0$ corresponds to midnight. Sketch the graph of f if $a = 0.5$ m.

50 **Tidal components** Refer to Exercise 49. The principal solar diurnal component may be approximated by

$$f(t) = a \cos\left(\frac{\pi}{12}t - \frac{7\pi}{12}\right).$$

Sketch the graph of f if $a = 0.2$ m.

51 **Hours of daylight in Fairbanks** If the formula for $D(t)$ in Example 12 is used for Fairbanks, Alaska, then $K \approx 12$. Sketch the graph of D in this case for $0 \le t \le 365$.

52 **Low temperature in Fairbanks** Based on years of weather data, the expected low temperature T (in °F) in Fairbanks, Alaska, can be approximated by

$$T = 36 \sin\left[\frac{2\pi}{365}(t - 101)\right] + 14,$$

where t is in days and $t = 0$ corresponds to January 1.

(a) Sketch the graph of T for $0 \le t \le 365$.

(b) Predict when the coldest day of the year will occur.

Exer. 53–56: Scientists sometimes use the formula

$$f(t) = a \sin(bt + c) + d$$

to simulate temperature variations during the day, with time t in hours, temperature $f(t)$ in °C, and $t = 0$ corresponding to midnight. Assume that $f(t)$ is decreasing at midnight.

(a) Determine values of a, b, c, and d that fit the information.

(b) Sketch the graph of f for $0 \le t \le 24$.

53 The high temperature is 10°C, and the low temperature of −10°C occurs at 4 A.M.

54 The temperature at midnight is 15°C, and the high and low temperatures are 20°C and 10°C.

55 The temperature varies between 10°C and 30°C, and the average temperature of 20°C first occurs at 9 A.M.

56 The high temperature of 28°C occurs at 2 P.M., and the average temperature of 20°C occurs 6 hours later.

6.6 Additional Trigonometric Graphs

Methods we developed in Section 6.5 for the sine and cosine can be applied to the other four trigonometric functions; however, there are several differences. Since the tangent, cotangent, secant, and cosecant functions have no largest values, the notion of amplitude has no meaning. Moreover, we do not refer to cycles. For some tangent and cotangent graphs, we begin by sketching the portion between successive vertical asymptotes and then repeat that pattern to the right and to the left.

The graph of $y = a \tan x$ for $a > 0$ can be obtained by stretching or compressing the graph of $y = \tan x$. If $a < 0$, then we also use a reflection about the x-axis. Since the tangent function has period π, it is sufficient to sketch the branch between the two successive vertical asymptotes $x = -\pi/2$ and $x = \pi/2$. The same pattern occurs to the right and to the left, as in the next example.

EXAMPLE 1 **Sketching the graph of an equation involving tan x**

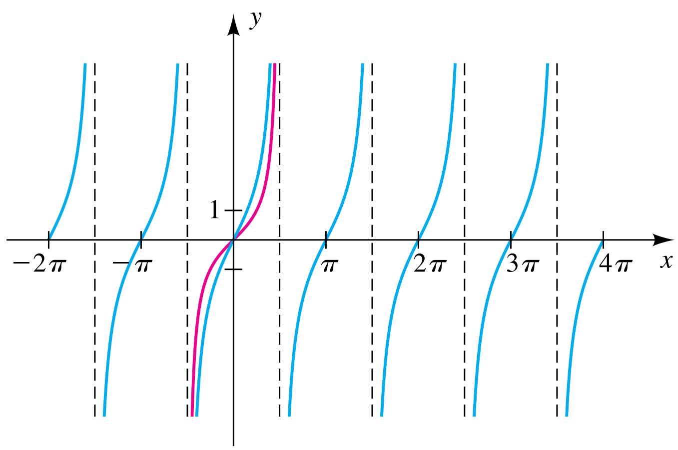
Sketch the graph of the equation:

(a) $y = 2 \tan x$ **(b)** $y = \frac{1}{2} \tan x$

SOLUTION We begin by sketching the graph of one branch of $y = \tan x$, as shown in red in Figures 1 and 2, between the vertical asymptotes $x = -\pi/2$ and $x = \pi/2$.

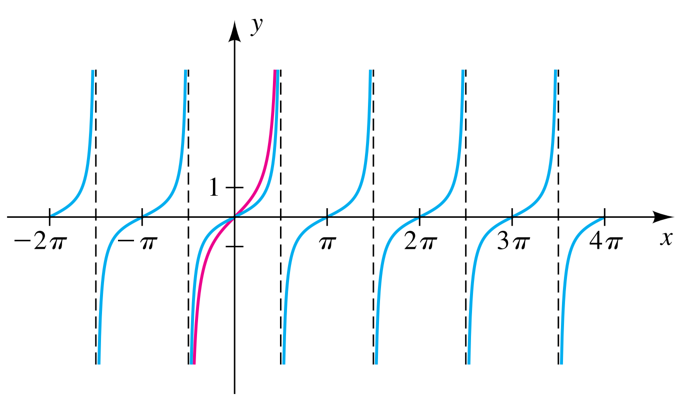
(a) For $y = 2 \tan x$, we multiply the y-coordinate of each point by 2 and then extend the resulting branch to the right and left, as shown in Figure 1.

Figure 1 $y = 2 \tan x$



(b) For $y = \frac{1}{2} \tan x$, we multiply the y-coordinates by $\frac{1}{2}$, obtaining the sketch in Figure 2.

Figure 2 $y = \frac{1}{2} \tan x$



The method used in Example 1 can be applied to other functions. Thus, to sketch the graph of $y = 3 \sec x$, we could first sketch the graph of one branch of $y = \sec x$ and then multiply the y-coordinate of each point by 3.

The next theorem is an analogue of the theorem on amplitudes, periods, and phase shifts stated in Section 6.5 for the sine and cosine functions.

Theorem on the Graph of $y = a \tan (bx + c)$

If $y = a \tan (bx + c)$ for nonzero real numbers a and b, then

(1) the period is $\dfrac{\pi}{|b|}$ and the phase shift is $-\dfrac{c}{b}$;

(2) successive vertical asymptotes for the graph of one branch may be found by solving the inequality

$$-\frac{\pi}{2} < bx + c < \frac{\pi}{2}.$$

EXAMPLE 2 Sketching the graph of an equation of the form $y = a \tan (bx + c)$

Find the period and sketch the graph of $y = \dfrac{1}{2} \tan \left(x + \dfrac{\pi}{4} \right)$.

SOLUTION The equation has the form given in the preceding theorem with $a = \frac{1}{2}$, $b = 1$, and $c = \pi/4$. Hence, by part (1), the period is given by $\pi/|b| = \pi/1 = \pi$.

As in part (2), to find successive vertical asymptotes we solve the following inequality:

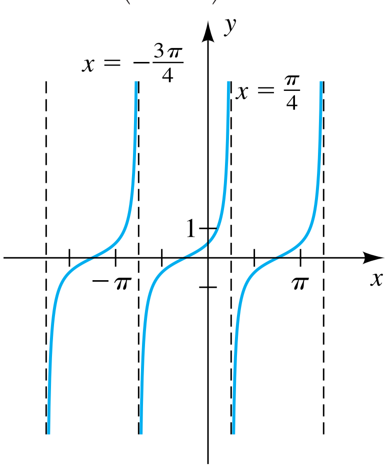
$$-\frac{\pi}{2} \le x + \frac{\pi}{4} \le \frac{\pi}{2}$$

$$-\frac{3\pi}{4} \le x \qquad \le \frac{\pi}{4} \qquad \text{subtract } \frac{\pi}{4}$$

Because $a = \frac{1}{2}$, the graph of the equation on the interval $[-3\pi/4, \pi/4]$ has the shape of the graph of $y = \frac{1}{2} \tan x$ (see Figure 2). Sketching that branch and extending it to the right and left gives us Figure 3.

Figure 3

$y = \dfrac{1}{2} \tan \left(x + \dfrac{\pi}{4} \right)$



Note that since $c = \pi/4$ and $b = 1$, the phase shift is $-c/b = -\pi/4$. Hence, the graph can also be obtained by shifting the graph of $y = \frac{1}{2} \tan x$ in Figure 2 to the left a distance $\pi/4$.

If $y = a \cot (bx + c)$, we have a situation similar to that stated in the previous theorem. The only difference is part (2). Since successive vertical asymptotes for the graph of $y = \cot x$ are $x = 0$ and $x = \pi$ (see Figure 19 in Section 6.3), we obtain successive vertical asymptotes for the graph of one branch of $y = a \cot (bx + c)$ by solving the inequality

$$0 < bx + c < \pi.$$

EXAMPLE 3 **Sketching the graph of an equation of the form $y = a \cot (bx + c)$**

Find the period and sketch the graph of $y = \cot\left(2x - \dfrac{\pi}{2}\right)$.

SOLUTION Using the usual notation, we see that $a = 1$, $b = 2$, and $c = -\pi/2$. The period is $\pi/|b| = \pi/2$. Hence, the graph repeats itself in intervals of length $\pi/2$.

As in the discussion preceding this example, to find two successive vertical asymptotes for the graph of one branch we solve the inequality:

$$0 \le 2x - \frac{\pi}{2} \le \pi$$

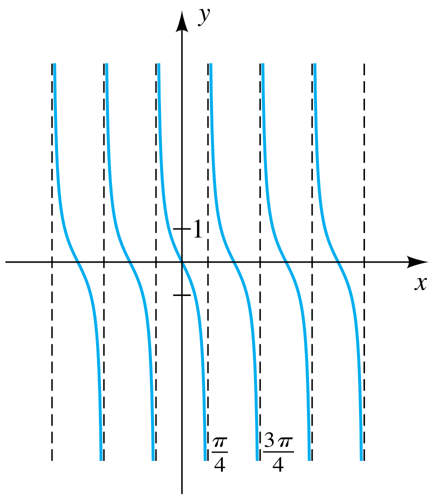
$$\frac{\pi}{2} \le 2x \le \frac{3\pi}{2} \quad \text{add } \frac{\pi}{2}$$

$$\frac{\pi}{4} \le x \le \frac{3\pi}{4} \quad \text{divide by 2}$$

Since a is positive, we sketch a cotangent-shaped branch on the interval $[\pi/4, 3\pi/4]$ and then repeat it to the right and left in intervals of length $\pi/2$, as shown in Figure 4.

Figure 4

$y = \cot\left(2x - \dfrac{\pi}{2}\right)$



Graphs involving the secant and cosecant functions can be obtained by using methods similar to those for the tangent and cotangent or by taking reciprocals of corresponding graphs of the cosine and sine functions.

EXAMPLE 4 **Sketching the graph of an equation of the form $y = a \sec (bx + c)$**

Sketch the graph of the equation:

(a) $y = \sec\left(x - \dfrac{\pi}{4}\right)$ **(b)** $y = 2\sec\left(x - \dfrac{\pi}{4}\right)$

SOLUTION

(a) The graph of $y = \sec x$ is sketched (without asymptotes) in red in Figure 5 on the next page. The graph of $y = \cos x$ is sketched in black; notice that the asymptotes of $y = \sec x$ correspond to the zeros of $y = \cos x$. We can obtain the graph of $y = \sec\left(x - \dfrac{\pi}{4}\right)$ by shifting the graph of $y = \sec x$ to the right a distance $\pi/4$, as shown in blue in Figure 5.

(b) We can sketch this graph by multiplying the y-coordinates of the graph in part (a) by 2. This gives us Figure 6 on the next page.

(continued)

Figure 5 $y = \sec\left(x - \dfrac{\pi}{4}\right)$

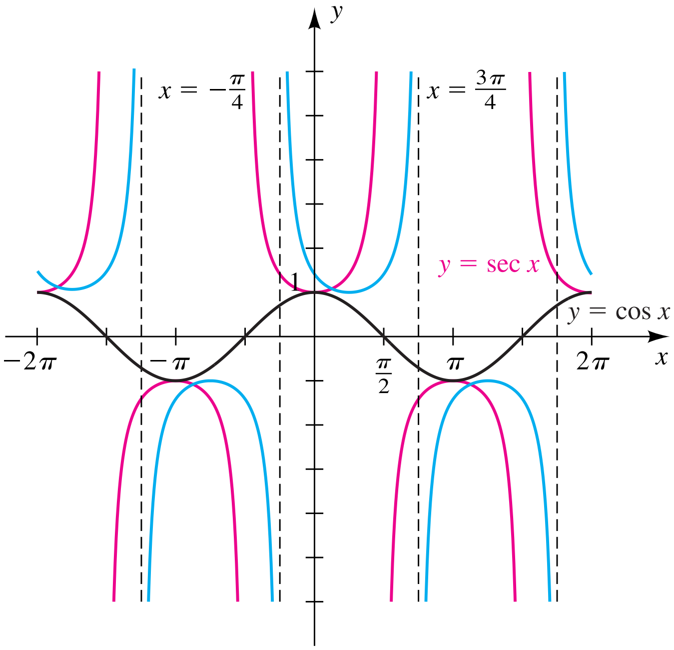
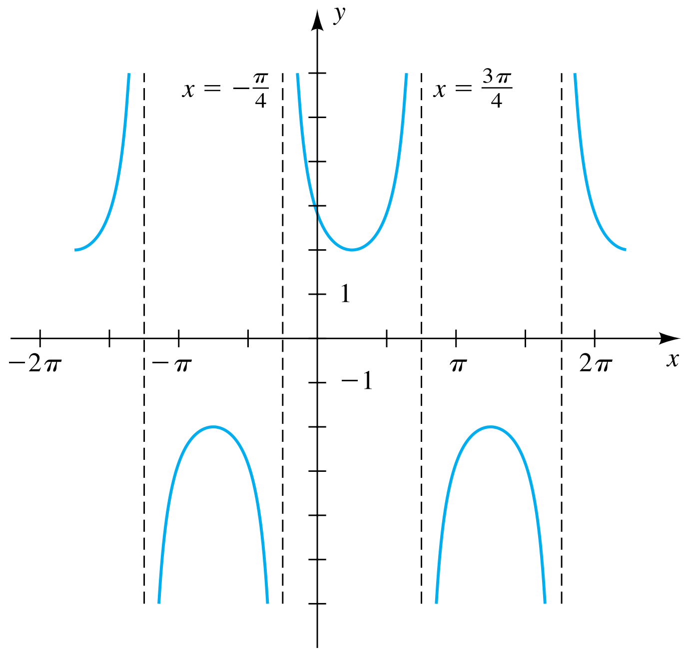


Figure 6 $y = 2\sec\left(x - \dfrac{\pi}{4}\right)$



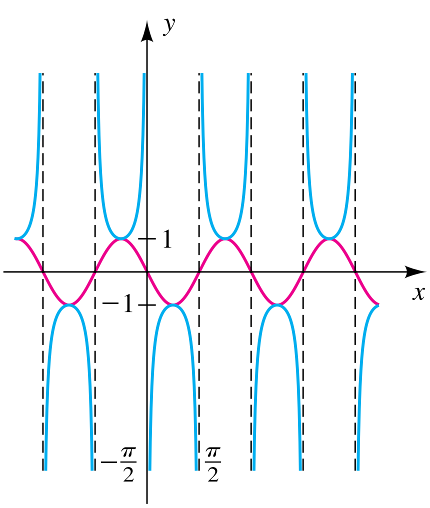
EXAMPLE 5 **Sketching the graph of an equation of the form $y = a \csc(bx + c)$**

Sketch the graph of $y = \csc(2x + \pi)$.

SOLUTION Since $\csc\theta = 1/\sin\theta$, we may write the given equation as

$$y = \frac{1}{\sin(2x + \pi)}.$$

Figure 7
$y = \csc(2x + \pi)$



Thus, we may obtain the graph of $y = \csc(2x + \pi)$ by finding the graph of $y = \sin(2x + \pi)$ and then taking the reciprocal of the y-coordinate of each point. Using $a = 1$, $b = 2$, and $c = \pi$, we see that the amplitude of $y = \sin(2x + \pi)$ is 1 and the period is $2\pi/|b| = 2\pi/2 = \pi$. To find an interval containing one cycle, we solve the inequality

$$\begin{aligned} 0 \le 2x + \pi &\le 2\pi \\ -\pi \le 2x \quad &\le \pi \\ -\frac{\pi}{2} \le x \quad &\le \frac{\pi}{2}. \end{aligned}$$

This leads to the graph in red in Figure 7. Taking reciprocals gives us the graph of $y = \csc(2x + \pi)$ shown in blue in the figure. Note that the zeros of the sine curve correspond to the asymptotes of the cosecant graph.

The next example involves the absolute value of a trigonometric function.

EXAMPLE 6 **Sketching the graph of an equation involving an absolute value**

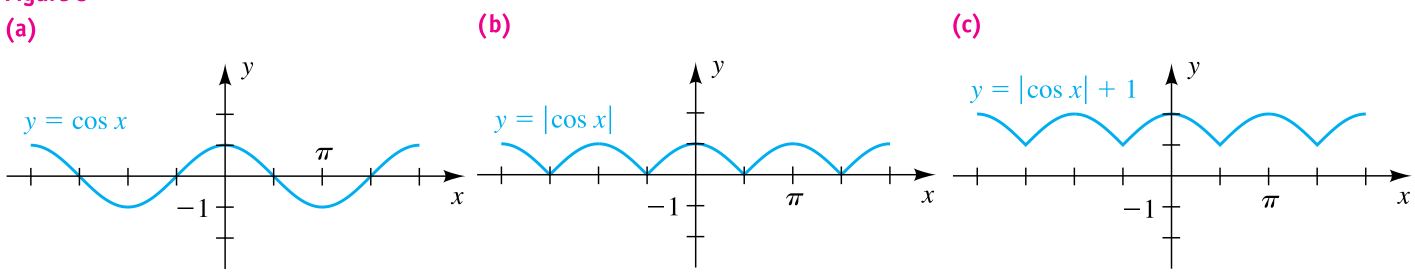
Sketch the graph of $y = |\cos x| + 1$.

SOLUTION We shall sketch the graph in three stages. First, we sketch the graph of $y = \cos x$, as in Figure 8(a).

Next, we obtain the graph of $y = |\cos x|$ by reflecting the negative y-coordinates in Figure 8(a) through the x-axis. This gives us Figure 8(b).

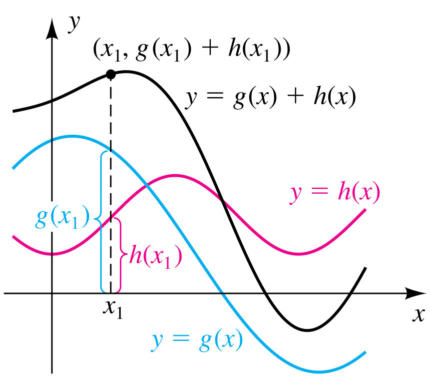
Finally, we vertically shift the graph in (b) upward 1 unit to obtain Figure 8(c).

Figure 8



We have used three separate graphs for clarity. In practice, we could sketch the graphs successively on one coordinate plane.

Figure 9



Mathematical applications often involve a function f that is a sum of two or more other functions. To illustrate, suppose

$$f(x) = g(x) + h(x),$$

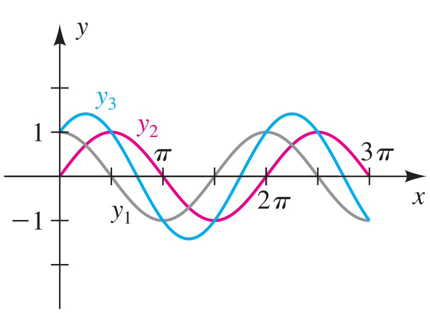
where f, g, and h have the same domain D. A technique known as **addition of y-coordinates** is sometimes used to sketch the graph of f. The method is illustrated in Figure 9, where for each x_1, the y-coordinate $f(x_1)$ of a point on the graph of f is the *sum* $g(x_1) + h(x_1)$ of the y-coordinates of points on the graphs of g and h. The graph of f is obtained by *graphically adding* a sufficient number of such y-coordinates.

It is sometimes useful to compare the graph of a sum of functions with the individual functions, as illustrated in the next example.

EXAMPLE 7 **Sketching the graph of a sum of two trigonometric functions**

Sketch the graph of $y_1 = \cos x$, $y_2 = \sin x$, and $y_3 = \cos x + \sin x$ on the same coordinate plane for $0 \le x \le 3\pi$.

Figure 10



SOLUTION Note that the graph of y_3 in Figure 10 intersects the graph of y_1 when $y_2 = 0$, and the graph of y_2 when $y_1 = 0$. The x-intercepts for y_3 correspond to the solutions of $y_2 = -y_1$. Finally, we see that the maximum and minimum values of y_3 occur when $y_1 = y_2$ (that is, when $x = \pi/4, 5\pi/4$, and $9\pi/4$). These y-values are

$$\sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2} \qquad \text{and} \qquad -\sqrt{2}/2 + \left(-\sqrt{2}/2\right) = -\sqrt{2}.$$

The graph of an equation of the form

$$y = f(x)\sin(ax + b) \qquad \text{or} \qquad y = f(x)\cos(ax + b),$$

where f is a function and a and b are real numbers, is called a **damped sine wave** or **damped cosine wave,** respectively, and $f(x)$ is called the **damping factor.** The next example illustrates a method for graphing such equations.

EXAMPLE 8 **Sketching the graph of a damped sine wave**

Sketch the graph of f if $f(x) = 2^{-x}\sin x$.

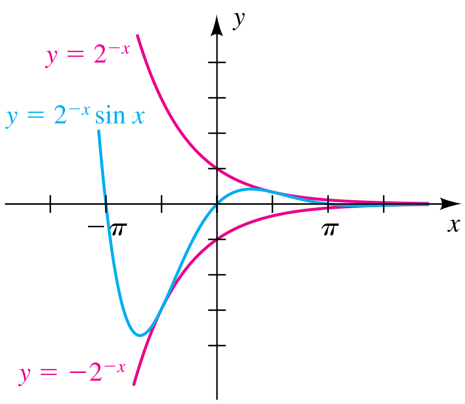
SOLUTION We first examine the absolute value of f:

$$\begin{aligned} |f(x)| &= |2^{-x}\sin x| && \text{absolute value of both sides} \\ &= |2^{-x}||\sin x| && |ab| = |a||b| \\ &\le |2^{-x}| \cdot 1 && |\sin x| \le 1 \\ |f(x)| &\le 2^{-x} && |2^{-x}| = 2^{-x} \text{ since } 2^{-x} > 0 \\ -2^{-x} \le f(x) &\le 2^{-x} && |x| \le a \iff -a \le x \le a \end{aligned}$$

The last inequality implies that the graph of f lies between the graphs of the equations $y = -2^{-x}$ and $y = 2^{-x}$. The graph of f will coincide with one of these graphs if $|\sin x| = 1$—that is, if $x = (\pi/2) + \pi n$ for some integer n.

Since $2^{-x} > 0$, the x-intercepts on the graph of f occur at $\sin x = 0$—that is, at $x = \pi n$. Because there are an infinite number of x-intercepts, this is an example of a function that intersects its horizontal asymptote an infinite number of times. With this information, we obtain the sketch shown in Figure 11.

Figure 11



The damping factor in Example 8 is 2^{-x}. By using different damping factors, we can obtain other compressed or expanded variations of sine waves. The analysis of such graphs is important in physics and engineering.

6.6 Exercises

Exer. 1–52: Find the period and sketch the graph of the equation. Show the asymptotes.

1 $y = 4\tan x$

2 $y = \frac{1}{4}\tan x$

3 $y = 3\cot x$

4 $y = \frac{1}{3}\cot x$

5 $y = 2\csc x$

6 $y = \frac{1}{2}\csc x$

7 $y = 3\sec x$

8 $y = \frac{1}{4}\sec x$

9 $y = \tan\left(x - \dfrac{\pi}{4}\right)$

10 $y = \tan\left(x + \dfrac{\pi}{2}\right)$

11 $y = \tan 2x$

12 $y = \tan \frac{1}{2}x$

13 $y = \tan \frac{1}{4}x$

14 $y = \tan 4x$

15 $y = 2\tan\left(2x + \dfrac{\pi}{2}\right)$

16 $y = \dfrac{1}{3}\tan\left(2x - \dfrac{\pi}{4}\right)$

17 $y = -\frac{1}{4}\tan\left(\frac{1}{2}x + \frac{\pi}{3}\right)$

18 $y = -3\tan\left(\frac{1}{3}x - \frac{\pi}{3}\right)$

19 $y = \cot\left(x - \frac{\pi}{2}\right)$

20 $y = \cot\left(x + \frac{\pi}{4}\right)$

21 $y = \cot 2x$

22 $y = \cot \frac{1}{2}x$

23 $y = \cot \frac{1}{3}x$

24 $y = \cot 3x$

25 $y = 2\cot\left(2x + \frac{\pi}{2}\right)$

26 $y = -\frac{1}{3}\cot(3x - \pi)$

27 $y = -\frac{1}{2}\cot\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

28 $y = 4\cot\left(\frac{1}{3}x - \frac{\pi}{6}\right)$

29 $y = \sec\left(x - \frac{\pi}{2}\right)$

30 $y = \sec\left(x - \frac{3\pi}{4}\right)$

31 $y = \sec 2x$

32 $y = \sec \frac{1}{2}x$

33 $y = \sec \frac{1}{3}x$

34 $y = \sec 3x$

35 $y = 2\sec\left(2x - \frac{\pi}{2}\right)$

36 $y = \frac{1}{2}\sec\left(2x - \frac{\pi}{2}\right)$

37 $y = -\frac{1}{3}\sec\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

38 $y = -3\sec\left(\frac{1}{3}x + \frac{\pi}{3}\right)$

39 $y = \csc\left(x - \frac{\pi}{2}\right)$

40 $y = \csc\left(x + \frac{3\pi}{4}\right)$

41 $y = \csc 2x$

42 $y = \csc \frac{1}{2}x$

43 $y = \csc \frac{1}{3}x$

44 $y = \csc 3x$

45 $y = 2\csc\left(2x + \frac{\pi}{2}\right)$

46 $y = -\frac{1}{2}\csc(2x - \pi)$

47 $y = -\frac{1}{4}\csc\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

48 $y = 4\csc\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

49 $y = \tan \frac{\pi}{2}x$

50 $y = \cot \pi x$

51 $y = \csc 2\pi x$

52 $y = \sec \frac{\pi}{8}x$

53 Find an equation using the cotangent function that has the same graph as $y = \tan x$.

54 Find an equation using the cosecant function that has the same graph as $y = \sec x$.

Exer. 55–60: Use the graph of a trigonometric function to aid in sketching the graph of the equation without plotting points.

55 $y = |\sin x|$

56 $y = |\cos x|$

57 $y = |\sin x| + 2$

58 $y = |\cos x| - 3$

59 $y = -|\cos x| + 1$

60 $y = -|\sin x| - 2$

Exer. 61–66: Sketch the graph of the equation.

61 $y = x + \cos x$

62 $y = x - \sin x$

63 $y = 2^{-x}\cos x$

64 $y = e^{x}\sin x$

65 $y = |x|\sin x$

66 $y = |x|\cos x$

67 **Radio signal intensity** Radio stations often have more than one broadcasting tower because federal guidelines do not usually permit a radio station to broadcast its signal in all directions with equal power. Since radio waves can travel over long distances, it is important to control their directional patterns so that radio stations do not interfere with one another. Suppose that a radio station has two broadcasting towers located along a north-south line, as shown in the figure. If the radio station is broadcasting at a wavelength λ and the

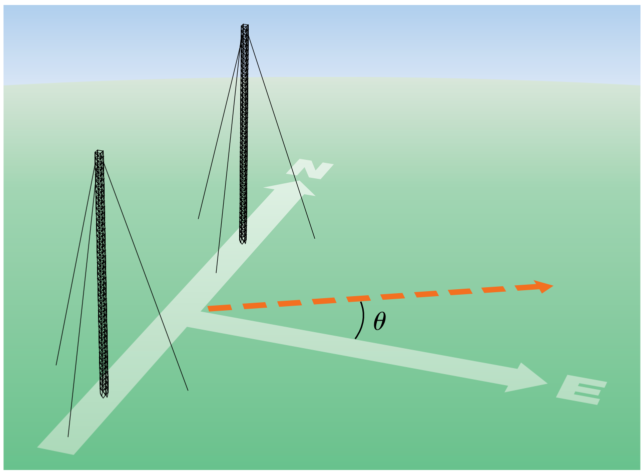
distance between the two radio towers is equal to $\frac{1}{2}\lambda$, then the intensity I of the signal in the direction θ is given by

$$I = \tfrac{1}{2}I_0[1 + \cos(\pi \sin \theta)],$$

where I_0 is the maximum intensity. Approximate I in terms of I_0 for each θ.

(a) $\theta = 0$ (b) $\theta = \pi/3$ (c) $\theta = \pi/7$

Exercise 67



68 Radio signal intensity Refer to Exercise 67. Determine the directions in which I has maximum or minimum values.

69 Earth's magnetic field The strength of Earth's magnetic field varies with the depth below the surface. The strength at depth z and time t can sometimes be approximated using the damped sine wave

$$S = A_0 e^{-\alpha z} \sin(kt - \alpha z),$$

where A_0, α, and k are constants.

(a) What is the damping factor?

(b) Find the phase shift at depth z_0.

(c) At what depth is the amplitude of the wave one-half the amplitude of the surface strength?

6.7 Applied Problems

Trigonometry was developed to help solve problems involving angles and lengths of sides of triangles. Problems of that type are no longer the most important applications; however, questions about triangles still arise in physical situations. When considering such questions in this section, we shall restrict our discussion to right triangles. Triangles that do not contain a right angle will be considered in Chapter 8.

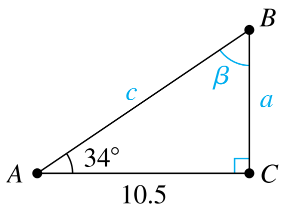
We shall often use the following notation. The vertices of a triangle will be denoted by A, B, and C; the angles at A, B, and C will be denoted by α, β, and γ, respectively; and the lengths of the sides opposite these angles by a, b, and c, respectively. The triangle itself will be referred to as *triangle ABC* (or denoted $\triangle ABC$). If a triangle is a right triangle and if one of the acute angles and a side are known or if two sides are given, then we may find the remaining parts by using the formulas in Section 6.2 that express the trigonometric functions as ratios of sides of a triangle. We can refer to the process of finding the remaining parts as **solving the triangle.**

In all examples *it is assumed that you know how to find trigonometric function values and angles by using either a calculator or results about special angles.*

EXAMPLE 1 **Solving a right triangle**

Solve $\triangle ABC$, given $\gamma = 90°$, $\alpha = 34°$, and $b = 10.5$.

Figure 1



SOLUTION Since the sum of the three interior angles in a triangle is 180°, we have $\alpha + \beta + \gamma = 180°$. Solving for the unknown angle β gives us

$$\beta = 180° - \alpha - \gamma = 180° - 34° - 90° = 56°.$$

Referring to Figure 1, we obtain

$$\tan 34° = \frac{a}{10.5} \qquad \tan \alpha = \frac{\text{opp}}{\text{adj}}$$

$$a = (10.5) \tan 34° \approx 7.1. \qquad \text{solve for } a\text{; approximate}$$

To find side c, we can use either the cosine or the secant function, as follows in (1) or (2), respectively:

(1) $$\cos 34° = \frac{10.5}{c} \qquad \cos \alpha = \frac{\text{adj}}{\text{hyp}}$$

$$c = \frac{10.5}{\cos 34°} \approx 12.7 \qquad \text{solve for } c\text{; approximate}$$

(2) $$\sec 34° = \frac{c}{10.5} \qquad \sec \alpha = \frac{\text{hyp}}{\text{adj}}$$

$$c = (10.5) \sec 34° \approx 12.7 \qquad \text{solve for } c\text{; approximate}$$

Homework Helper

Organizing your work in a table makes it easy to see what parts remain to be found. Here are some snapshots of what a typical table might look like for Example 1.

After finding β:

Angles	*Opposite sides*
$\alpha = 34°$	a
$\beta = 56°$	$b = 10.5$
$\gamma = 90°$	c

After finding a:

Angles	*Opposite sides*
$\alpha = 34°$	$a \approx 7.1$
$\beta = 56°$	$b = 10.5$
$\gamma = 90°$	c

After finding c:

Angles	*Opposite sides*
$\alpha = 34°$	$a \approx 7.1$
$\beta = 56°$	$b = 10.5$
$\gamma = 90°$	$c \approx 12.7$

As illustrated in Example 1, when working with triangles, we usually round off answers. One reason for doing so is that in most applications the lengths of sides of triangles and measures of angles are found by mechanical devices and hence are only approximations to the exact values. Consequently, a number such as 10.5 in Example 1 is assumed to have been rounded off to the nearest tenth. We cannot expect more accuracy in the calculated values for the remaining sides, and therefore they should also be rounded off to the nearest tenth.

In finding angles, answers should be rounded off as indicated in the following table.

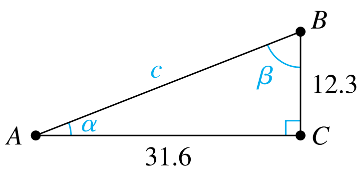
Number of significant figures for sides	Round off degree measure of angles to the nearest
2	1°
3	0.1°, or 10′
4	0.01°, or 1′

Justification of this table requires a careful analysis of problems that involve approximate data.

EXAMPLE 2 **Solving a right triangle**

Solve $\triangle ABC$, given $\gamma = 90°$, $a = 12.3$, and $b = 31.6$.

Figure 2



SOLUTION Referring to the triangle illustrated in Figure 2 gives us

$$\tan \alpha = \frac{12.3}{31.6}.$$

Since the sides are given with three significant figures, the rule stated in the preceding table tells us that α should be rounded off to the nearest 0.1°, or the nearest multiple of 10′. Using the degree mode on a calculator, we have

$$\alpha = \tan^{-1}\frac{12.3}{31.6} \approx 21.3° \qquad \text{or, equivalently,} \qquad \alpha \approx 21°20'.$$

Since α and β are complementary angles,

$$\beta = 90° - \alpha \approx 90° - 21.3° = 68.7°.$$

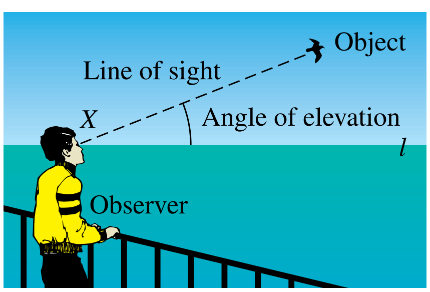
The only remaining part to find is c. We could use several relationships involving c to determine its value. Among these are

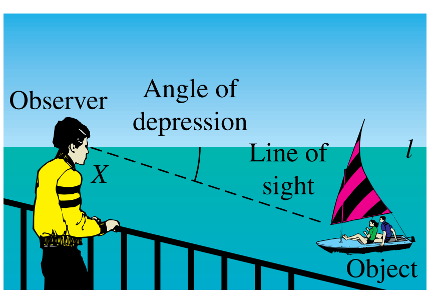
$$\cos \alpha = \frac{31.6}{c}, \qquad \sec \beta = \frac{c}{12.3}, \qquad \text{and} \qquad a^2 + b^2 = c^2.$$

Whenever possible, it is best to use a relationship that involves only given information, since it doesn't depend on any previously calculated value. Hence, with $a = 12.3$ and $b = 31.6$, we have

$$c = \sqrt{a^2 + b^2} = \sqrt{(12.3)^2 + (31.6)^2} = \sqrt{1149.85} \approx 33.9.$$

Figure 3





As illustrated in Figure 3, if an observer at point X sights an object, then the angle that the line of sight makes with the horizontal line l is the **angle of elevation** of the object, if the object is above the horizontal line, or the **angle of depression** of the object, if the object is below the horizontal line. We use this terminology in the next two examples.

EXAMPLE 3 **Using an angle of elevation**

From a point on level ground 135 feet from the base of a tower, the angle of elevation of the top of the tower is 57°20′. Approximate the height of the tower.

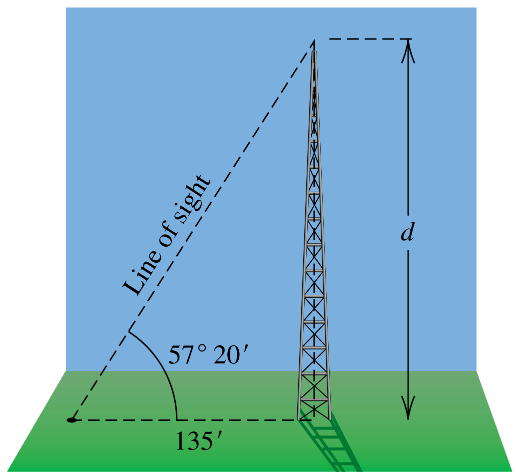
SOLUTION If we let d denote the height of the tower, then the given facts are represented by the triangle in Figure 4. Referring to the figure, we obtain

$$\tan 57°20' = \frac{d}{135} \qquad \tan 57°20' = \frac{\text{opp}}{\text{adj}}$$

$$d = 135 \tan 57°20' \approx 211. \qquad \text{solve for } d\text{; approximate}$$

The tower is approximately 211 feet high.

Figure 4

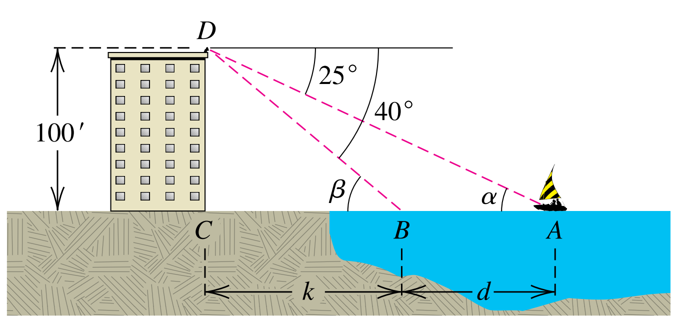


EXAMPLE 4 **Using angles of depression**

From the top of a building that overlooks an ocean, an observer watches a boat sailing directly toward the building. If the observer is 100 feet above sea level and if the angle of depression of the boat changes from 25° to 40° during the period of observation, approximate the distance that the boat travels.

SOLUTION As in Figure 5, let A and B be the positions of the boat that correspond to the 25° and 40° angles, respectively. Suppose that the observer is at point D and that C is the point 100 feet directly below. Let d denote the distance the boat travels, and let k denote the distance from B to C. If α and β

Figure 5



(continued)

denote angles DAC and DBC, respectively, then it follows from geometry (alternate interior angles) that $\alpha = 25°$ and $\beta = 40°$.

From triangle BCD:

$$\cot \beta = \cot 40° = \frac{k}{100} \qquad \cot \beta = \frac{\text{adj}}{\text{opp}}$$

$$k = 100 \cot 40° \qquad \text{solve for } k$$

From triangle DAC:

$$\cot \alpha = \cot 25° = \frac{d + k}{100} \qquad \cot \alpha = \frac{\text{adj}}{\text{opp}}$$

$$d + k = 100 \cot 25° \qquad \text{multiply by lcd}$$

$$d = 100 \cot 25° - k \qquad \text{solve for } d$$

$$= 100 \cot 25° - 100 \cot 40° \qquad k = 100 \cot 40°$$

$$= 100(\cot 25° - \cot 40°) \qquad \text{factor out 100}$$

$$\approx 100(2.145 - 1.192) \approx 95 \qquad \text{approximate}$$

Note that $d = \overline{AC} - \overline{BC}$, and if we use tan instead of cot, we get the equivalent equation

$$d = \frac{100}{\tan 25°} - \frac{100}{\tan 40°}.$$

Hence, the boat travels approximately 95 feet.

In certain navigation or surveying problems, the **direction,** or **bearing,** from a point P to a point Q is specified by stating the acute angle that segment PQ makes with the north-south line through P. We also state whether Q is north or south and east or west of P. Figure 6 illustrates four possibilities. The bearing from P to Q_1 is 25° east of north and is denoted by N25°E. We also refer to the **direction** N25°E, meaning the direction from P to Q_1. The bearings from P to Q_2, to Q_3, and to Q_4 are represented in a similar manner in the figure. Note that when this notation is used for bearings or directions, N or S always appears to the *left* of the angle and W or E to the *right.*

Figure 6

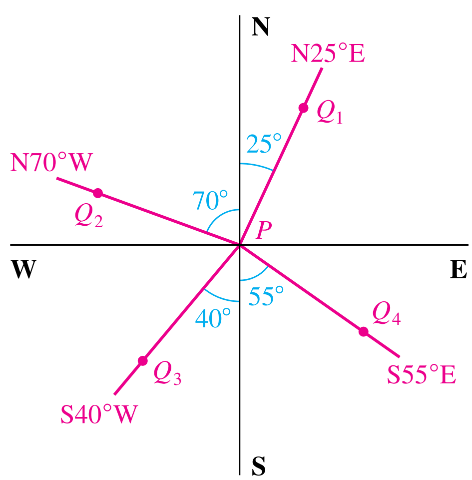
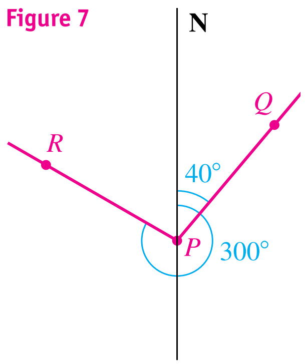


Figure 7

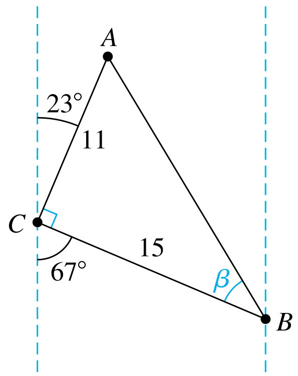


In air navigation, directions and bearings are specified by measuring from the north in a *clockwise* direction. In this case, a positive measure is assigned to the angle instead of the negative measure to which we are accustomed for clockwise rotations. Referring to Figure 7, we see that the direction of PQ is 40° and the direction of PR is 300°.

EXAMPLE 5 Using bearings

Two ships leave port at the same time, one ship sailing in the direction N23°E at a speed of 11 mi/hr and the second ship sailing in the direction S67°E at 15 mi/hr. Approximate the bearing from the second ship to the first, one hour later.

Figure 8



SOLUTION The sketch in Figure 8 indicates the positions of the first and second ships at points A and B, respectively, after one hour. Point C represents the port. We wish to find the bearing from B to A. Note that

$$\angle ACB = 180° - 23° - 67° = 90°,$$

and hence triangle ACB is a right triangle. Thus,

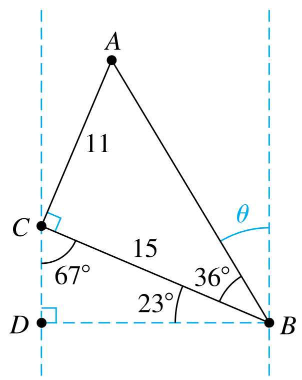
$$\tan\beta = \frac{11}{15} \qquad \tan\beta = \frac{\text{opp}}{\text{adj}}$$

$$\beta = \tan^{-1}\tfrac{11}{15} \approx 36°. \qquad \text{solve for } \beta\text{; approximate}$$

We have rounded β to the nearest degree because the sides of the triangles are given with two significant figures.

Referring to Figure 9, we obtain the following:

Figure 9



$$\angle CBD = 90° - \angle BCD = 90° - 67° = 23°$$
$$\angle ABD = \angle ABC + \angle CBD \approx 36° + 23° = 59°$$
$$\theta = 90° - \angle ABD \approx 90° - 59° = 31°$$

Thus, the bearing from B to A is approximately N31°W.

Trigonometric functions are useful in the investigation of vibratory or oscillatory motion, such as the motion of a particle in a vibrating guitar string or a spring that has been compressed or elongated and then released to oscillate back and forth. The fundamental type of particle displacement in these illustrations is *harmonic motion.*

Definition of Simple Harmonic Motion

A point moving on a coordinate line is in **simple harmonic motion** if its distance d from the origin at time t is given by either

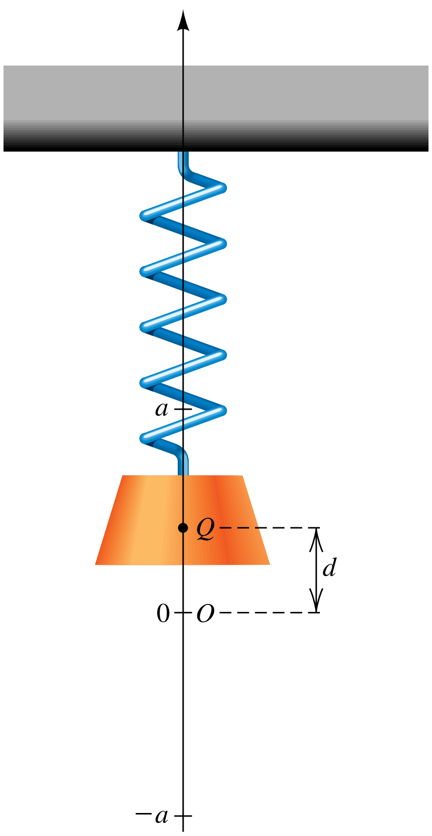
$$d = a\cos\omega t \qquad \text{or} \qquad d = a\sin\omega t,$$

where a and ω are constants, with $\omega > 0$.

In the preceding definition, the **amplitude** of the motion is the maximum displacement $|a|$ of the point from the origin. The **period** is the time $2\pi/\omega$ required for one complete oscillation. The reciprocal of the period, $\omega/(2\pi)$, is the number of oscillations per unit of time and is called the **frequency.**

A physical interpretation of simple harmonic motion can be obtained by considering a spring with an attached weight that is oscillating vertically relative to a coordinate line, as illustrated in Figure 10. The number d represents the coordinate of a fixed point Q in the weight, and we assume that the amplitude a of the motion is constant. In this case no frictional force is retarding the motion. If friction is present, then the amplitude decreases with time, and the motion is said to be *damped.*

Figure 10



EXAMPLE 6 Describing harmonic motion

Suppose that the oscillation of the weight shown in Figure 10 is given by

$$d = 10 \cos\left(\frac{\pi}{6}t\right),$$

with t measured in seconds and d in centimeters. Discuss the motion of the weight.

SOLUTION By definition, the motion is simple harmonic with amplitude $a = 10$ cm. Since $\omega = \pi/6$, we obtain the following:

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/6} = 12$$

Thus, in 12 seconds the weight makes one complete oscillation. The frequency is $\frac{1}{12}$, which means that one-twelfth of an oscillation takes place each second. The following table indicates the position of Q at various times.

t	0	1	2	3	4	5	6
$\frac{\pi}{6}t$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\cos\left(\frac{\pi}{6}t\right)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1
d	10	$5\sqrt{3} \approx 8.7$	5	0	-5	$-5\sqrt{3} \approx -8.7$	-10

The initial position of Q is 10 centimeters above the origin O. It moves downward, gaining speed until it reaches O. Note that Q travels approximately $10 - 8.7 = 1.3$ cm during the first second, $8.7 - 5 = 3.7$ cm during the next second, and $5 - 0 = 5$ cm during the third second. It then slows down until it reaches a point 10 centimeters below O at the end of 6 seconds. The direction of motion is then reversed, and the weight moves upward, gaining speed until it reaches O. Once it reaches O, it slows down until it returns to its original position at the end of 12 seconds. The direction of motion is then reversed again, and the same pattern is repeated indefinitely.

6.7 Exercises

Exer. 1–8: Given the indicated parts of triangle ABC with $\gamma = 90°$, find the exact values of the remaining parts.

1 $\alpha = 30°$, $b = 20$

2 $\beta = 45°$, $b = 35$

3 $\beta = 45°$, $c = 30$

4 $\alpha = 60°$, $c = 6$

5 $a = 5$, $b = 5$

6 $a = 4\sqrt{3}$, $c = 8$

7 $b = 5\sqrt{3}$, $c = 10\sqrt{3}$

8 $b = 7\sqrt{2}$, $c = 14$

Exer. 9–16: Given the indicated parts of triangle ABC with $\gamma = 90°$, approximate the remaining parts.

9 $\alpha = 37°$, $b = 24$

10 $\beta = 64°20'$, $a = 20.1$

11 $\beta = 71°51'$, $b = 240.0$

12 $\alpha = 31°10'$, $a = 510$

13 $a = 25$, $b = 45$

14 $a = 31$, $b = 9.0$

15 $c = 5.8$, $b = 2.1$

16 $a = 0.42$, $c = 0.68$

Exer. 17–24: Given the indicated parts of triangle ABC with $\gamma = 90°$, express the third part in terms of the first two.

17 α, c; b

18 β, c; b

19 β, b; a

20 α, b; a

21 α, a; c

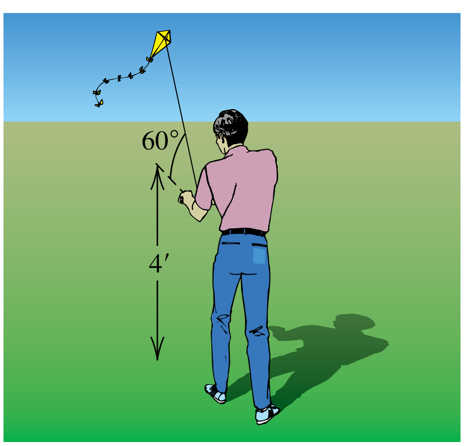
22 β, a; c

23 a, c; b

24 a, b; c

25 **Height of a kite** A person flying a kite holds the string 4 feet above ground level. The string of the kite is taut and makes an angle of 60° with the horizontal (see the figure). Approximate the height of the kite above level ground if 500 feet of string is payed out.

Exercise 25



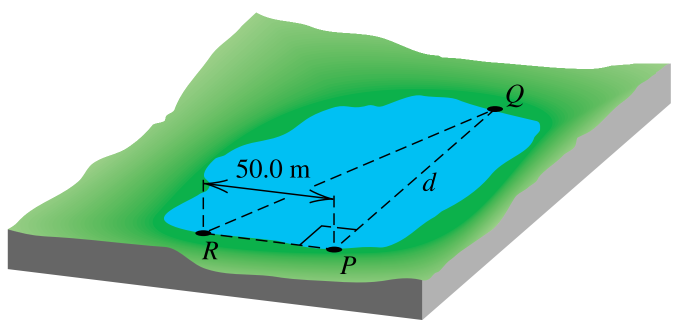
26 **Surveying** From a point 15 meters above level ground, a surveyor measures the angle of depression of an object on the ground at 68°. Approximate the distance from the object to the point on the ground directly beneath the surveyor.

27 **Airplane landing** A pilot, flying at an altitude of 5000 feet, wishes to approach the numbers on a runway at an angle of 10°. Approximate, to the nearest 100 feet, the distance from the airplane to the numbers at the beginning of the descent.

28 **Radio antenna** A guy wire is attached to the top of a radio antenna and to a point on horizontal ground that is 40.0 meters from the base of the antenna. If the wire makes an angle of 58°20′ with the ground, approximate the length of the wire.

29 **Surveying** To find the distance d between two points P and Q on opposite shores of a lake, a surveyor locates a point R that is 50.0 meters from P such that RP is perpendicular to PQ, as shown in the figure. Next, using a transit, the surveyor measures angle PRQ as 72°40′. Find d.

Exercise 29

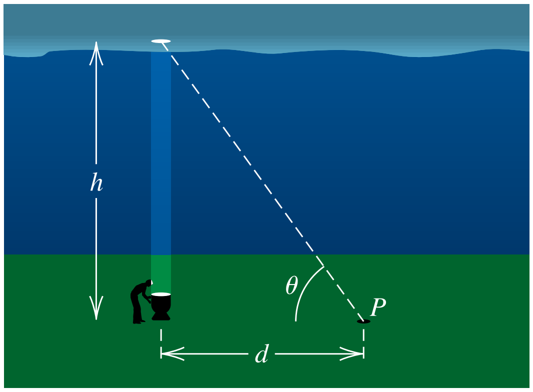


30 **Meteorological calculations** To measure the height h of a cloud cover, a meteorology student directs a spotlight vertically upward from the ground. From a point P on level ground that is d meters from the spotlight, the angle of elevation θ of the light image on the clouds is then measured (see the figure on the next page).

(a) Express h in terms of d and θ.

(b) Approximate h if $d = 1000$ m and $\theta = 59°$.

Exercise 30



31 Altitude of a rocket A rocket is fired at sea level and climbs at a constant angle of 75° through a distance of 10,000 feet. Approximate its altitude to the nearest foot.

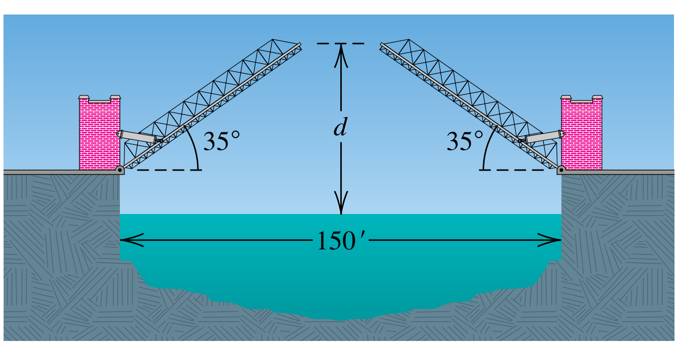
32 Airplane takeoff An airplane takes off at a 10° angle and travels at the rate of 250 ft/sec. Approximately how long does it take the airplane to reach an altitude of 15,000 feet?

33 Designing a drawbridge A drawbridge is 150 feet long when stretched across a river. As shown in the figure, the two sections of the bridge can be rotated upward through an angle of 35°.

(a) If the water level is 15 feet below the closed bridge, find the distance d between the end of a section and the water level when the bridge is fully open.

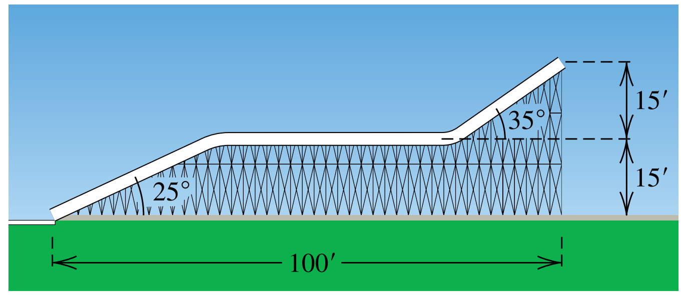
(b) Approximately how far apart are the ends of the two sections when the bridge is fully opened, as shown in the figure?

Exercise 33



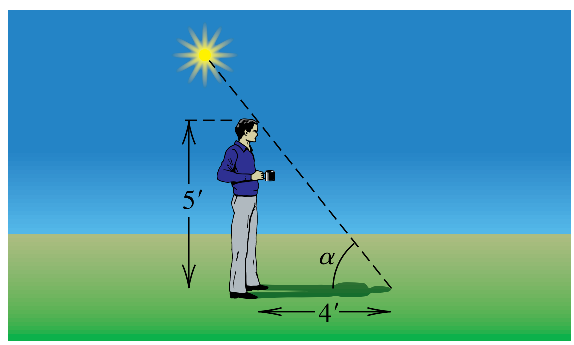
34 Designing a water slide Shown in the figure is part of a design for a water slide. Find the total length of the slide to the nearest foot.

Exercise 34



35 Sun's elevation Approximate the angle of elevation α of the sun if a person 5.0 feet tall casts a shadow 4.0 feet long on level ground (see the figure).

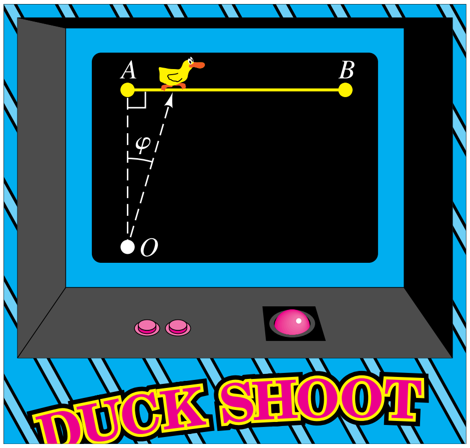
Exercise 35



36 Constructing a ramp A builder wishes to construct a ramp 24 feet long that rises to a height of 5.0 feet above level ground. Approximate the angle that the ramp should make with the horizontal.

37 Video game Shown in the figure is the screen for a simple video arcade game in which ducks move from A to B at the rate of 7 cm/sec. Bullets fired from point O travel 25 cm/sec. If a player shoots as soon as a duck appears at A, at which angle φ should the gun be aimed in order to score a direct hit?

Exercise 37

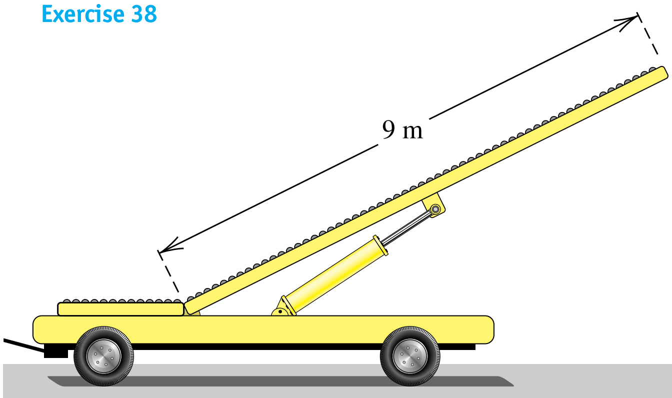


38 Conveyor belt A conveyor belt 9 meters long can be hydraulically rotated up to an angle of 40° to unload cargo from airplanes (see the figure).

(a) Find, to the nearest degree, the angle through which the conveyor belt should be rotated up to reach a door that is 4 meters above the platform supporting the belt.

(b) Approximate the maximum height above the platform that the belt can reach.

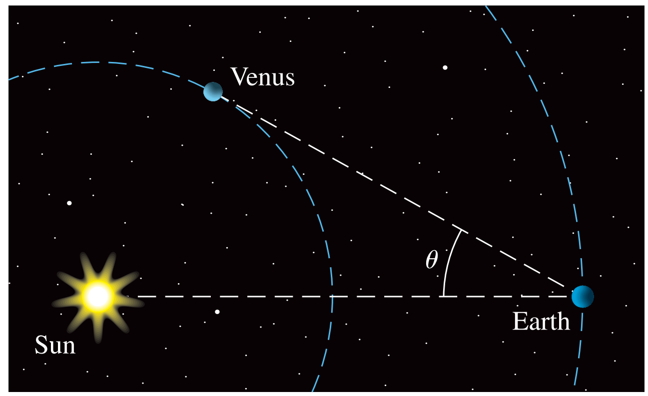
Exercise 38



39 Tallest structure The tallest man-made structure in the world is a television transmitting tower located near Mayville, North Dakota. From a distance of 1 mile on level ground, its angle of elevation is 21°20′24″. Determine its height to the nearest foot.

40 Elongation of Venus The *elongation* of the planet Venus is defined to be the angle θ determined by the sun, Earth, and Venus, as shown in the figure. Maximum elongation of Venus occurs when Earth is at its minimum distance D_e from the sun and Venus is at its maximum distance D_v from the sun. If $D_e = 91{,}500{,}000$ mi and $D_v = 68{,}000{,}000$ mi, approximate the maximum elongation θ_{max} of Venus. Assume that the orbit of Venus is circular.

Exercise 40

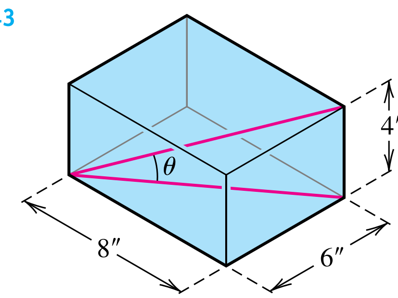


41 The Pentagon's ground area The Pentagon is the largest office building in the world in terms of ground area. The perimeter of the building has the shape of a regular pentagon with each side of length 921 feet. Find the area enclosed by the perimeter of the building.

42 A regular octagon is inscribed in a circle of radius 12.0 centimeters. Approximate the perimeter of the octagon.

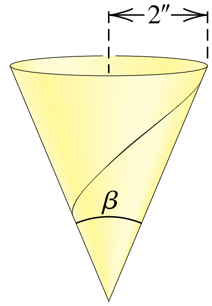
43 A rectangular box has dimensions 8″ × 6″ × 4″. Approximate, to the nearest tenth of a degree, the angle θ formed by a diagonal of the base and the diagonal of the box, as shown in the figure.

Exercise 43



44 Volume of a conical cup A conical paper cup has a radius of 2 inches. Approximate, to the nearest degree, the angle β (see the figure) so that the cone will have a volume of 20 in^3.

Exercise 44



45 **Height of a tower** From a point P on level ground, the angle of elevation of the top of a tower is 26°50′. From a point 25.0 meters closer to the tower and on the same line with P and the base of the tower, the angle of elevation of the top is 53°30′. Approximate the height of the tower.

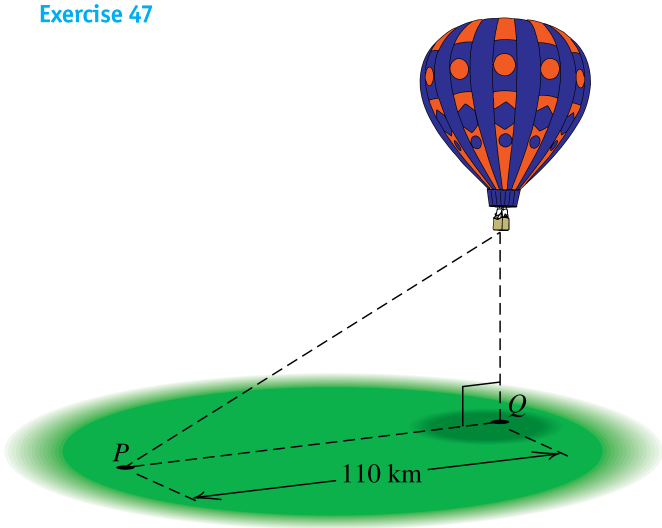
46 **Ladder calculations** A ladder 20 feet long leans against the side of a building, and the angle between the ladder and the building is 22°.

(a) Approximate the distance from the bottom of the ladder to the building.

(b) If the distance from the bottom of the ladder to the building is increased by 3.0 feet, approximately how far does the top of the ladder move down the building?

47 **Ascent of a hot-air balloon** As a hot-air balloon rises vertically, its angle of elevation from a point P on level ground 110 kilometers from the point Q directly underneath the balloon changes from 19°20′ to 31°50′ (see the figure). Approximately how far does the balloon rise during this period?

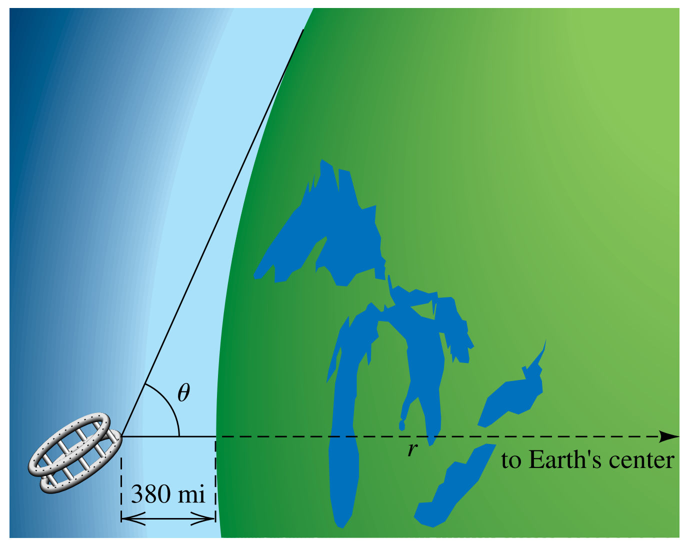
Exercise 47



48 **Height of a building** From a point A that is 8.20 meters above level ground, the angle of elevation of the top of a building is 31°20′ and the angle of depression of the base of the building is 12°50′. Approximate the height of the building.

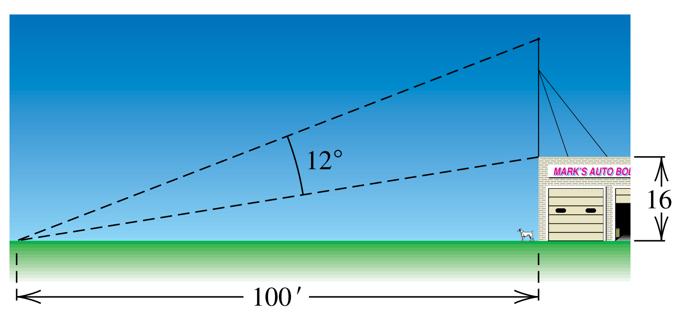
49 **Radius of Earth** A spacelab circles Earth at an altitude of 380 miles. When an astronaut views the horizon of Earth, the angle θ shown in the figure is 65.8°. Use this information to estimate the radius of Earth.

Exercise 49



50 **Length of an antenna** A CB antenna is located on the top of a garage that is 16 feet tall. From a point on level ground that is 100 feet from a point directly below the antenna, the antenna subtends an angle of 12°, as shown in the figure. Approximate the length of the antenna.

Exercise 50



51 **Speed of an airplane** An airplane flying at an altitude of 10,000 feet passes directly over a fixed object on the ground. One minute later, the angle of depression of the object is 42°. Approximate the speed of the airplane to the nearest mile per hour.

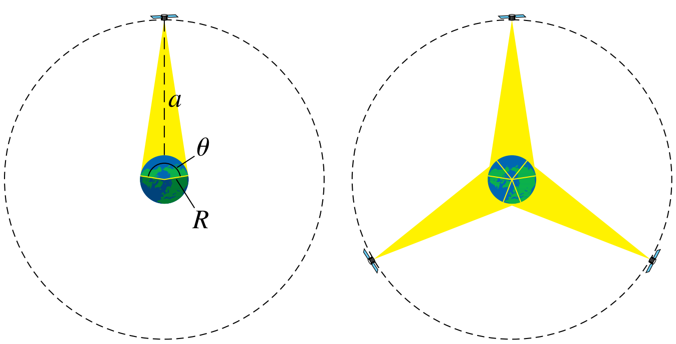
52 **Height of a mountain** A motorist, traveling along a level highway at a speed of 60 km/hr directly toward a mountain, observes that between 1:00 P.M. and 1:10 P.M. the angle of elevation of the top of the mountain changes from 10° to 70°. Approximate the height of the mountain.

53 **Communications satellite** Shown in the left part of the figure is a communications satellite with an equatorial orbit—that is, a nearly circular orbit in the plane determined by Earth's equator. If the satellite circles Earth at an altitude of $a = 22{,}300$ mi, its speed is the same as the rotational speed of Earth; to an observer on the equator, the satellite appears to be stationary—that is, its orbit is synchronous.

(a) Using $R = 4000$ mi for the radius of Earth, determine the percentage of the equator that is within signal range of such a satellite.

(b) As shown in the right part of the figure, three satellites are equally spaced in equatorial synchronous orbits. Use the value of θ obtained in part (a) to explain why all points on the equator are within signal range of at least one of the three satellites.

Exercise 53

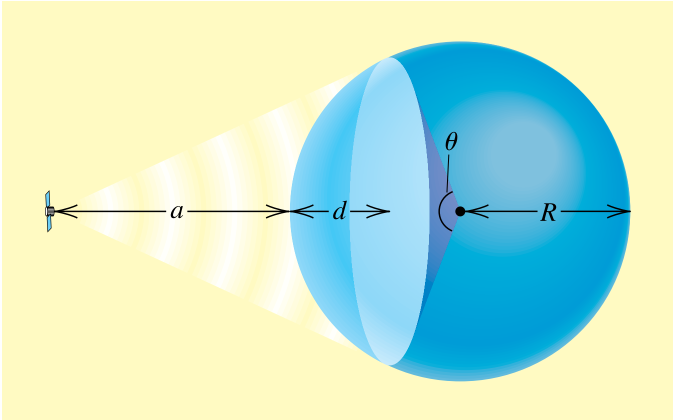


54 **Communications satellite** Refer to Exercise 53. Shown in the figure is the area served by a communications satellite circling a planet of radius R at an altitude a. The portion of the planet's surface within range of the satellite is a spherical cap of depth d and surface area $A = 2\pi Rd$.

(a) Express d in terms of R and θ.

(b) Estimate the percentage of the planet's surface that is within signal range of a single satellite in equatorial synchronous orbit.

Exercise 54



55 **Height of a kite** Generalize Exercise 25 to the case where the angle is α, the number of feet of string payed out is d, and the end of the string is held c feet above the ground. Express the height h of the kite in terms of α, d, and c.

56 **Surveying** Generalize Exercise 26 to the case where the point is d meters above level ground and the angle of depression is α. Express the distance x in terms of d and α.

57 **Height of a tower** Generalize Exercise 45 to the case where the first angle is α, the second angle is β, and the distance between the two points is d. Express the height h of the tower in terms of d, α, and β.

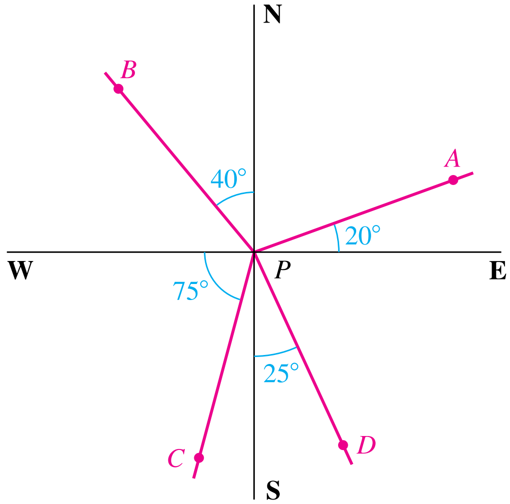
58 Generalize Exercise 42 to the case of an n-sided polygon inscribed in a circle of radius r. Express the perimeter P in terms of n and r.

59 **Ascent of a hot-air balloon** Generalize Exercise 47 to the case where the distance from P to Q is d kilometers and the angle of elevation changes from α to β.

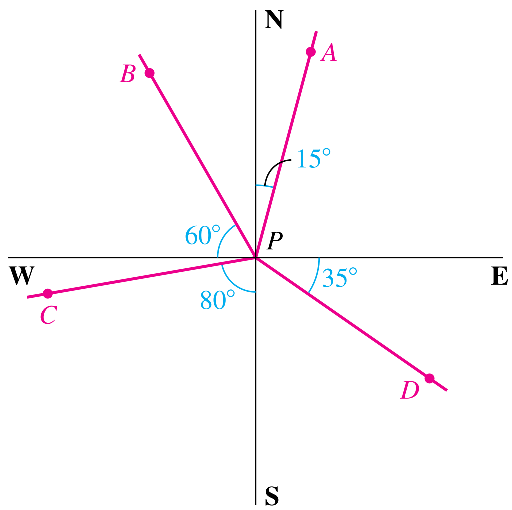
60 **Height of a building** Generalize Exercise 48 to the case where point A is d meters above ground and the angles of elevation and depression are α and β, respectively. Express the height h of the building in terms of d, α, and β.

Exer. 61–62: Find the bearing from P to each of the points A, B, C, and D.

61



62



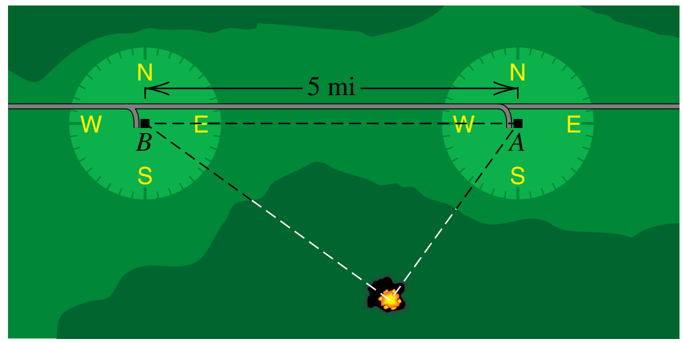
63 **Ship's bearings** A ship leaves port at 1:00 P.M. and sails in the direction N34°W at a rate of 24 mi/hr. Another ship leaves port at 1:30 P.M. and sails in the direction N56°E at a rate of 18 mi/hr.

(a) Approximately how far apart are the ships at 3:00 P.M.?

(b) What is the bearing, to the nearest degree, from the first ship to the second?

64 **Pinpointing a forest fire** From an observation point A, a forest ranger sights a fire in the direction S35°50′W (see the figure). From a point B, 5 miles due west of A, another ranger sights the same fire in the direction S54°10′E. Approximate, to the nearest tenth of a mile, the distance of the fire from A.

Exercise 64



65 **Airplane flight** An airplane flying at a speed of 360 mi/hr flies from a point A in the direction 137° for 30 minutes and then flies in the direction 227° for 45 minutes. Approximate, to the nearest mile, the distance from the airplane to A.

66 **Airplane flight plan** An airplane flying at a speed of 400 mi/hr flies from a point A in the direction 153° for 1 hour and then flies in the direction 63° for 1 hour.

(a) In what direction does the plane need to fly in order to get back to point A?

(b) How long will it take to get back to point A?

Exer. 67–70: The formula specifies the position of a point P that is moving harmonically on a vertical axis, where t is in seconds and d is in centimeters. Determine the amplitude, period, and frequency, and describe the motion of the point during one complete oscillation (starting at $t = 0$).

67 $d = 10 \sin 6\pi t$

68 $d = \frac{1}{3} \cos \frac{\pi}{4} t$

69 $d = 4 \cos \frac{3\pi}{2} t$

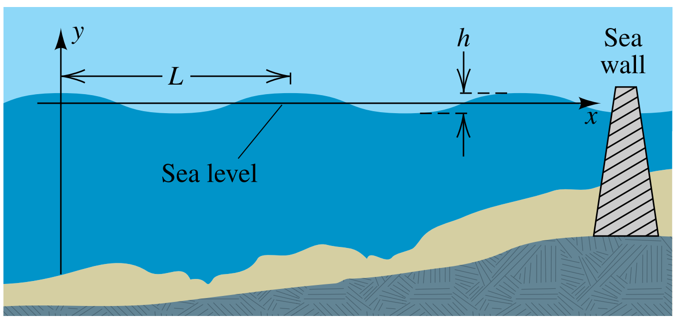
70 $d = 6 \sin \frac{2\pi}{3} t$

71 A point P in simple harmonic motion has a period of 3 seconds and an amplitude of 5 centimeters. Express the motion of P by means of an equation of the form $d = a \cos \omega t$.

72 A point P in simple harmonic motion has a frequency of $\frac{1}{2}$ oscillation per minute and an amplitude of 4 feet. Express the motion of P by means of an equation of the form $d = a \sin \omega t$.

73 **Tsunamis** A tsunami is a tidal wave caused by an earthquake beneath the sea. These waves can be more than 100 feet in height and can travel at great speeds. Engineers sometimes represent such waves by trigonometric expressions of the form $y = a \cos bt$ and use these representations to estimate the effectiveness of sea walls. Suppose that a wave has height $h = 50$ ft and period 30 minutes and is traveling at the rate of 180 ft/sec.

Exercise 73



(a) Let (x, y) be a point on the wave represented in the figure. Express y as a function of t if $y = 25$ ft when $t = 0$.

(b) The wave length L is the distance between two successive crests of the wave. Approximate L in feet.

74 **Some Hawaiian tsunamis** For an interval of 45 minutes, the tsunamis near Hawaii caused by the Chilean earthquake of 1960 could be modeled by the equation $y = 8 \sin \dfrac{\pi}{6} t$, where y is in feet and t is in minutes.

(a) Find the amplitude and period of the waves.

(b) If the distance from one crest of the wave to the next was 21 kilometers, what was the velocity of the wave? (Tidal waves can have velocities of more than 700 km/hr in deep sea water.)

CHAPTER 6 REVIEW EXERCISES

1 Find the radian measure that corresponds to each degree measure: $330°, 405°, -150°, 240°, 36°$.

2 Find the degree measure that corresponds to each radian measure: $\dfrac{9\pi}{2}, -\dfrac{2\pi}{3}, \dfrac{7\pi}{4}, 5\pi, \dfrac{\pi}{5}$.

3 A central angle θ is subtended by an arc 20 centimeters long on a circle of radius 2 meters.

(a) Find the radian measure of θ.

(b) Find the area of the sector determined by θ.

4 (a) Find the length of the arc that subtends an angle of measure $70°$ on a circle of diameter 15 centimeters.

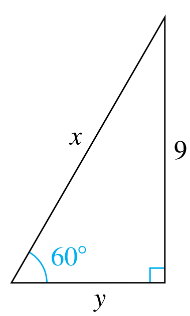
(b) Find the area of the sector in part (a).

5 **Angular speed of phonograph records** Two types of phonograph records, LP albums and singles, have diameters of 12 inches and 7 inches, respectively. The album rotates at a rate of $33\frac{1}{3}$ rpm, and the single rotates at 45 rpm. Find the angular speed (in radians per minute) of the album and of the single.

6 **Linear speed on phonograph records** Using the information in Exercise 5, find the linear speed (in ft/min) of a point on the circumference of the album and of the single.

Exer. 7–8: Find the exact values of x and y.

7



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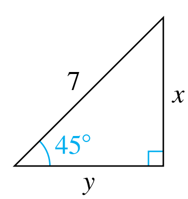


Exer. 9–10: Use fundamental identities to write the first expression in terms of the second, for any acute angle θ.

9 $\tan\theta$, $\sec\theta$

10 $\cot\theta$, $\csc\theta$

Exer. 11–20: Verify the identity by transforming the left-hand side into the right-hand side.

11 $\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$

12 $\cos\theta(\tan\theta + \cot\theta) = \csc\theta$

13 $(\cos^2\theta - 1)(\tan^2\theta + 1) = 1 - \sec^2\theta$

14 $\dfrac{\sec\theta - \cos\theta}{\tan\theta} = \dfrac{\tan\theta}{\sec\theta}$

15 $\dfrac{1 + \tan^2\theta}{\tan^2\theta} = \csc^2\theta$

16 $\dfrac{\sec\theta + \csc\theta}{\sec\theta - \csc\theta} = \dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}$

17 $\dfrac{\cot\theta - 1}{1 - \tan\theta} = \cot\theta$

18 $\dfrac{1 + \sec\theta}{\tan\theta + \sin\theta} = \csc\theta$

19 $\dfrac{\tan(-\theta) + \cot(-\theta)}{\tan\theta} = -\csc^2\theta$

20 $-\dfrac{1}{\csc(-\theta)} - \dfrac{\cot(-\theta)}{\sec(-\theta)} = \csc\theta$

21 If θ is an acute angle of a right triangle and if the adjacent side and hypotenuse have lengths 4 and 7, respectively, find the values of the trigonometric functions of θ.

22 Whenever possible, find the exact values of the trigonometric functions of θ if θ is in standard position and satisfies the stated condition.

(a) The point $(30, -40)$ is on the terminal side of θ.

(b) The terminal side of θ is in quadrant II and is parallel to the line $2x + 3y + 6 = 0$.

(c) The terminal side of θ is on the negative y-axis.

23 Find the quadrant containing θ if θ is in standard position.

(a) $\sec\theta < 0$ and $\sin\theta > 0$

(b) $\cot\theta > 0$ and $\csc\theta < 0$

(c) $\cos\theta > 0$ and $\tan\theta < 0$

24 Find the exact values of the remaining trigonometric functions if

(a) $\sin\theta = -\frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

(b) $\csc\theta = \dfrac{\sqrt{13}}{2}$ and $\cot\theta = -\dfrac{3}{2}$

Exer. 25–26: $P(t)$ denotes the point on the unit circle U that corresponds to the real number t.

25 Find the rectangular coordinates of $P(7\pi)$, $P(-5\pi/2)$, $P(9\pi/2)$, $P(-3\pi/4)$, $P(18\pi)$, and $P(\pi/6)$.

26 If $P(t)$ has coordinates $\left(-\frac{3}{5}, -\frac{4}{5}\right)$, find the coordinates of $P(t + 3\pi)$, $P(t - \pi)$, $P(-t)$, and $P(2\pi - t)$.

27 (a) Find the reference angle for each radian measure: $\dfrac{5\pi}{4}, -\dfrac{5\pi}{6}, -\dfrac{9\pi}{8}$.

(b) Find the reference angle for each degree measure: $245°$, $137°$, $892°$.

28 Without the use of a calculator, find the exact values of the trigonometric functions corresponding to each real number, whenever possible.

(a) $\dfrac{9\pi}{2}$ (b) $-\dfrac{5\pi}{4}$ (c) 0 (d) $\dfrac{11\pi}{6}$

29 Find the exact value.

(a) $\cos 225°$ (b) $\tan 150°$ (c) $\sin\left(-\dfrac{\pi}{6}\right)$

(d) $\sec\dfrac{4\pi}{3}$ (e) $\cot\dfrac{7\pi}{4}$ (f) $\csc 300°$

30 If $\sin\theta = -0.7604$ and $\sec\theta$ is positive, approximate θ to the nearest $0.1°$ for $0° \le \theta < 360°$.

31 If $\tan\theta = 2.7381$, approximate θ to the nearest 0.0001 radian for $0 \le \theta < 2\pi$.

32 If $\sec\theta = 1.6403$, approximate θ to the nearest $0.01°$ for $0° \le \theta < 360°$.

Exer. 33–40: Find the amplitude and period and sketch the graph of the equation.

33 $y = 5\cos x$

34 $y = \frac{2}{3}\sin x$

35 $y = \frac{1}{3}\sin 3x$

36 $y = -\frac{1}{2}\cos\frac{1}{3}x$

37 $y = -3\cos\frac{1}{2}x$

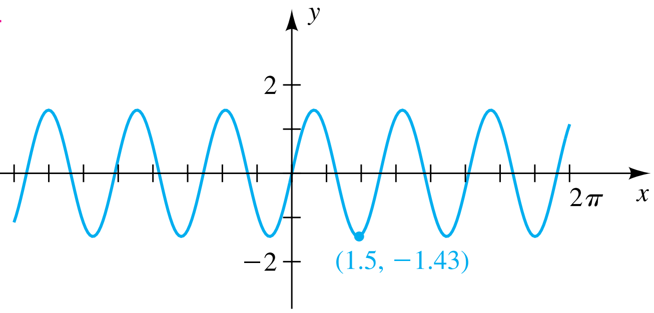
38 $y = 4\sin 2x$

39 $y = 2\sin\pi x$

40 $y = 4\cos\dfrac{\pi}{2}x - 2$

Exer. 41–44: The graph of an equation is shown in the figure. (a) Find the amplitude and period. (b) Express the equation in the form $y = a \sin bx$ or in the form $y = a \cos bx$.

41



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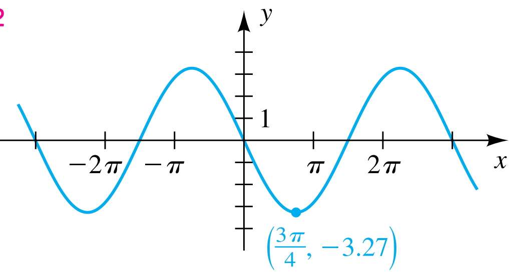

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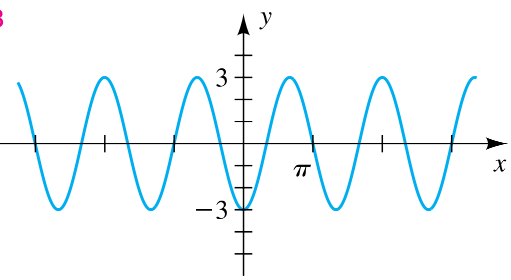

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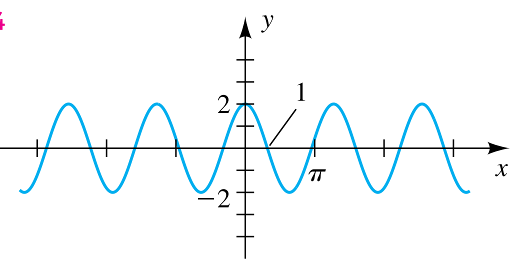


Exer. 45–56: Sketch the graph of the equation.

45 $y = 2 \sin\left(x - \frac{2\pi}{3}\right)$

46 $y = -3 \sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

47 $y = -4 \cos\left(x + \frac{\pi}{6}\right)$

48 $y = 5 \cos\left(2x + \frac{\pi}{2}\right)$

49 $y = 2 \tan\left(\frac{1}{2}x - \pi\right)$

50 $y = -3 \tan\left(2x + \frac{\pi}{3}\right)$

51 $y = -4 \cot\left(2x - \frac{\pi}{2}\right)$

52 $y = 2 \cot\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

53 $y = \sec\left(\frac{1}{2}x + \pi\right)$

54 $y = \sec\left(2x - \frac{\pi}{2}\right)$

55 $y = \csc\left(2x - \frac{\pi}{4}\right)$

56 $y = \csc\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

Exer. 57–60: Given the indicated parts of triangle ABC with $\gamma = 90°$, approximate the remaining parts.

57 $\beta = 60°$, $b = 40$

58 $\alpha = 54°40'$, $b = 220$

59 $a = 62$, $b = 25$

60 $a = 9.0$, $c = 41$

61 **Airplane propeller** The length of the largest airplane propeller ever used was 22 feet 7.5 inches. The plane was powered by four engines that turned the propeller at 545 revolutions per minute.

(a) What was the angular speed of the propeller in radians per second?

(b) Approximately how fast (in mi/hr) did the tip of the propeller travel along the circle it generated?

62 **The Eiffel Tower** When the top of the Eiffel Tower is viewed at a distance of 200 feet from the base, the angle of elevation is 79.2°. Estimate the height of the tower.

63 **Lasers and velocities** Lasers are used to accurately measure velocities of objects. Laser light produces an oscillating electromagnetic field E with a constant frequency f that can be described by

$$E = E_0 \cos(2\pi f t).$$

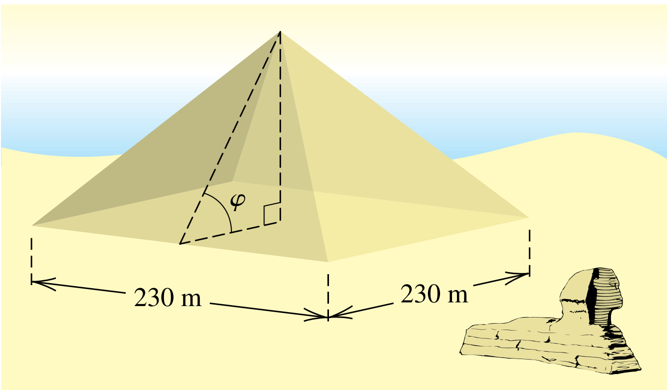
If a laser beam is pointed at an object moving toward the laser, light will be reflected toward the laser at a slightly higher frequency, in much the same way as a train whistle sounds higher when it is moving toward you. If Δf is this change in frequency and v is the object's velocity, then the equation

$$\Delta f = \frac{2fv}{c}$$

can be used to determine v, where $c = 186{,}000$ mi/sec is the velocity of the light. Approximate the velocity v of an object if $\Delta f = 10^8$ and $f = 10^{14}$.

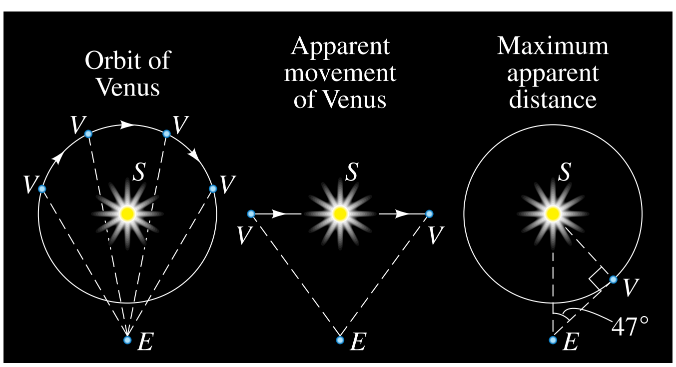
64 The Great Pyramid The Great Pyramid of Egypt is 147 meters high, with a square base of side 230 meters (see the figure). Approximate, to the nearest degree, the angle φ formed when an observer stands at the midpoint of one of the sides and views the apex of the pyramid.

Exercise 64



65 Venus When viewed from Earth over a period of time, the planet Venus appears to move back and forth along a line segment with the sun at its midpoint (see the figure). If *ES* is approximately 92,900,000 miles, then the maximum apparent distance of Venus from the sun occurs when angle *SEV* is approximately 47°. Assume that the orbit of Venus is circular and estimate the distance of Venus from the sun.

Exercise 65



66 Surveying From a point 233 feet above level ground, a surveyor measures the angle of depression of an object on the ground as 17°. Approximate the distance from the object to the point on the ground directly beneath the surveyor.

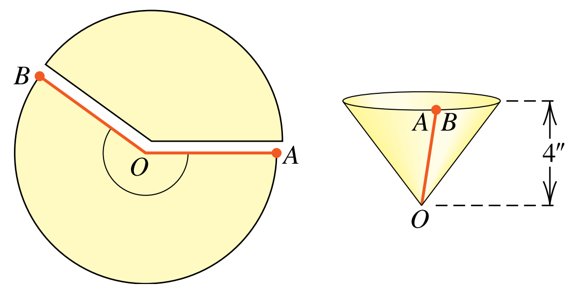
67 Ladder calculations A ladder 16 feet long leans against the side of a building, and the angle between the ladder and the building is 25°.

(a) Approximate the distance from the bottom of the ladder to the building.

(b) If the distance from the bottom of the ladder to the building is decreased by 1.5 feet, approximately how far does the top of the ladder move up the building?

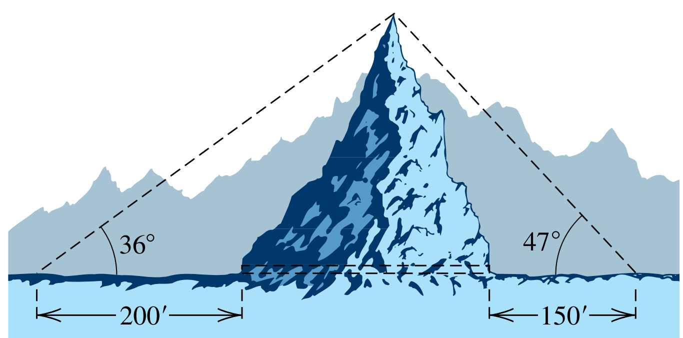
68 Constructing a conical cup A conical paper cup is constructed by removing a sector from a circle of radius 5 inches and attaching edge *OA* to *OB* (see the figure). Find angle *AOB* so that the cup has a depth of 4 inches.

Exercise 68



69 Length of a tunnel A tunnel for a new highway is to be cut through a mountain that is 260 feet high. At a distance of 200 feet from the base of the mountain, the angle of elevation is 36° (see the figure). From a distance of 150 feet on the other side, the angle of elevation is 47°. Approximate the length of the tunnel to the nearest foot.

Exercise 69

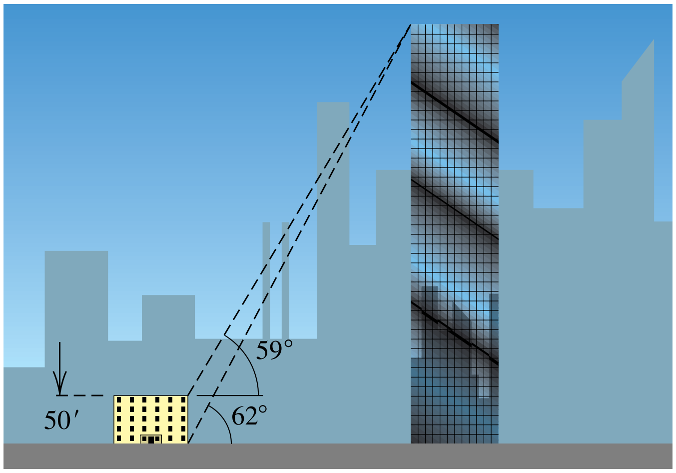


70 Height of a skyscraper When a certain skyscraper is viewed from the top of a building 50 feet tall, the angle of elevation is 59° (see the figure). When viewed from the street next to the shorter building, the angle of elevation is 62°.

(a) Approximately how far apart are the two structures?

(b) Approximate the height of the skyscraper to the nearest tenth of a foot.

Exercise 70



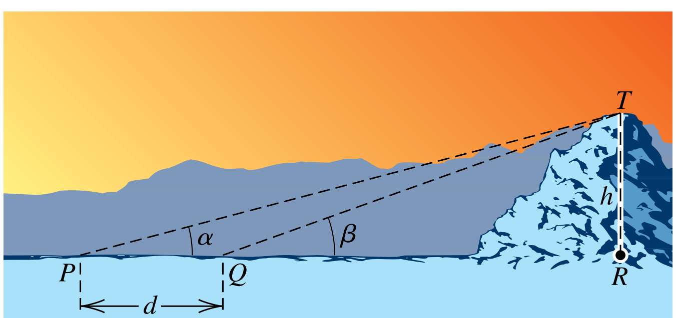
71 **Height of a mountain** When a mountaintop is viewed from the point P shown in the figure, the angle of elevation is α. From a point Q, which is d miles closer to the mountain, the angle of elevation increases to β.

(a) Show that the height h of the mountain is given by

$$h = \frac{d}{\cot\,\alpha - \cot\,\beta}.$$

(b) If $d = 2$ mi, $\alpha = 15°$, and $\beta = 20°$, approximate the height of the mountain.

Exercise 71

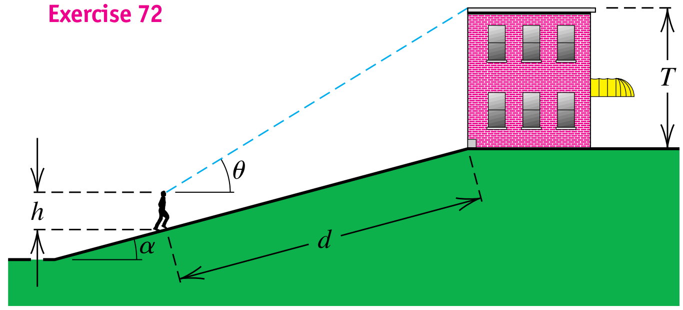


72 **Height of a building** An observer of height h stands on an incline at a distance d from the base of a building of height T, as shown in the figure. The angle of elevation from the observer to the top of the building is θ, and the incline makes an angle of α with the horizontal.

(a) Express T in terms of h, d, α, and θ.

(b) If $h = 6$ ft, $d = 50$ ft, $\alpha = 15°$, and $\theta = 31.4°$, estimate the height of the building.

Exercise 72



73 **Illuminance** A spotlight with intensity 5000 candles is located 15 feet above a stage. If the spotlight is rotated through an angle θ as shown in the figure, the illuminance E (in foot-candles) in the lighted area of the stage is given by

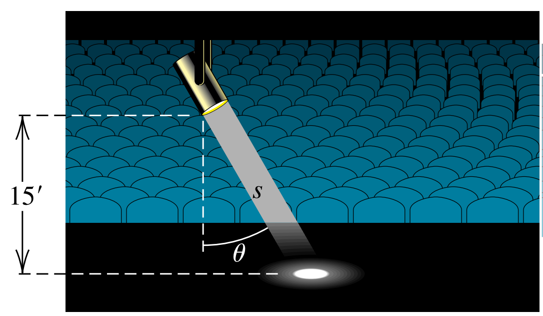
$$E = \frac{5000\,\cos\,\theta}{s^2},$$

where s is the distance (in feet) that the light must travel.

(a) Find the illuminance if the spotlight is rotated through an angle of 30°.

(b) The maximum illuminance occurs when $\theta = 0°$. For what value of θ is the illuminance one-half the maximum value?

Exercise 73



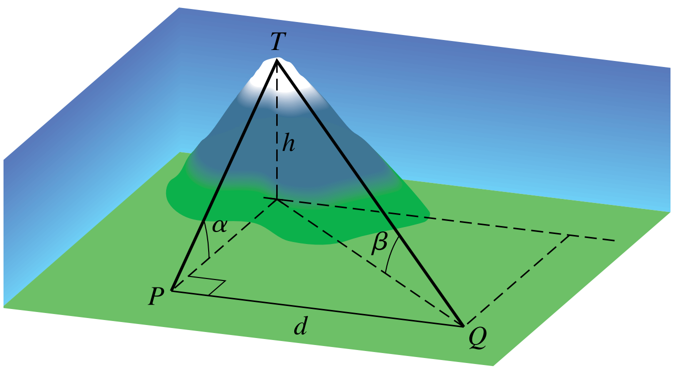
74 **Height of a mountain** If a mountaintop is viewed from a point P due south of the mountain, the angle of elevation is α (see the figure). If viewed from a point Q that is d miles east of P, the angle of elevation is β.

(a) Show that the height h of the mountain is given by

$$h = \frac{d\,\sin\,\alpha\,\sin\,\beta}{\sqrt{\sin^2\,\alpha - \sin^2\,\beta}}.$$

(b) If $\alpha = 30°$, $\beta = 20°$, and $d = 10$ mi, approximate h to the nearest hundredth of a mile.

Exercise 74

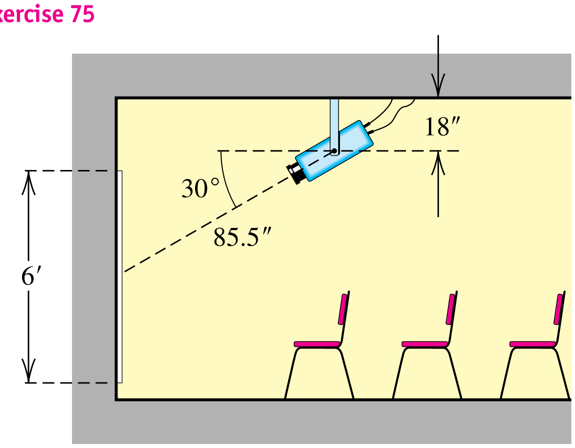


75 Mounting a projection unit The manufacturer of a computerized projection system recommends that a projection unit be mounted on the ceiling as shown in the figure. The distance from the end of the mounting bracket to the center of the screen is 85.5 inches, and the angle of depression is 30°.

(a) If the thickness of the screen is disregarded, how far from the wall should the bracket be mounted?

(b) If the bracket is 18 inches long and the screen is 6 feet high, determine the distance from the ceiling to the top edge of the screen.

Exercise 75

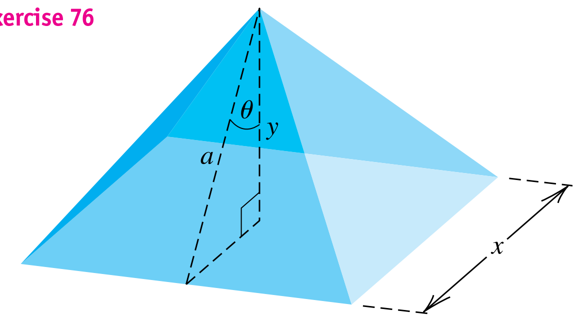


76 Pyramid relationships A pyramid has a square base and congruent triangular faces. Let θ be the angle that the altitude a of a triangular face makes with the altitude y of the pyramid, and let x be the length of a side (see the figure).

(a) Express the total surface area S of the four faces in terms of a and θ.

(b) The volume V of the pyramid equals one-third the area of the base times the altitude. Express V in terms of a and θ.

Exercise 76

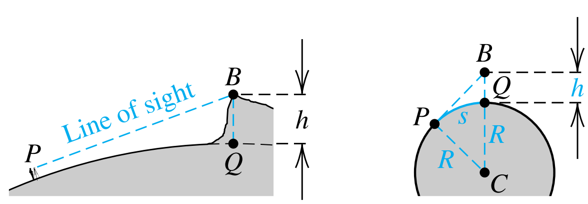


77 Surveying a bluff A surveyor, using a transit, sights the edge B of a bluff, as shown in the left part of the figure (not drawn to scale). Because of the curvature of Earth, the true elevation h of the bluff is larger than that measured by the surveyor. A cross-sectional schematic view of Earth is shown in the right part of the figure.

(a) If s is the length of arc PQ and R is the distance from P to the center C of Earth, express h in terms of R and s.

(b) If $R = 4000$ mi and $s = 50$ mi, estimate the elevation of the bluff in feet.

Exercise 77

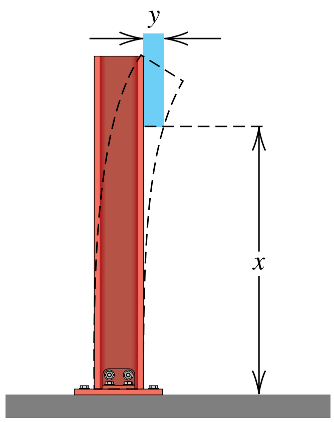


78 Earthquake response To simulate the response of a structure to an earthquake, an engineer must choose a shape for the initial displacement of the beams in the building. When the beam has length L feet and the maximum displacement is a feet, the equation

$$y = a - a \cos \frac{\pi}{2L} x$$

has been used by engineers to estimate the displacement y (see the figure). If $a = 1$ and $L = 10$, sketch the graph of the equation for $0 \le x \le 10$.

Exercise 78



79 **Circadian rhythms** The variation in body temperature is an example of a circadian rhythm, a cycle of a biological process that repeats itself approximately every 24 hours. Body temperature is highest about 5 P.M. and lowest at 5 A.M. Let y denote the body temperature (in °F), and let $t = 0$ correspond to midnight. If the low and high body temperatures are 98.3° and 98.9°, respectively, find an equation having the form $y = 98.6 + a \sin(bt + c)$ that fits this information.

80 **Temperature variation in Ottawa** The annual variation in temperature T (in °C) in Ottawa, Canada, may be approximated by

$$T(t) = 15.8 \sin\left[\frac{\pi}{6}(t - 3)\right] + 5,$$

where t is the time in months and $t = 0$ corresponds to January 1.

(a) Sketch the graph of T for $0 \le t \le 12$.

(b) Find the highest temperature of the year and the date on which it occurs.

81 **Water demand** A reservoir supplies water to a community. During the summer months, the demand $D(t)$ for water (in ft^3/day) is given by

$$D(t) = 2000 \sin \frac{\pi}{90} t + 4000,$$

where t is time in days and $t = 0$ corresponds to the beginning of summer.

(a) Sketch the graph of D for $0 \le t \le 90$.

(b) When is the demand for water the greatest?

82 **Bobbing cork** A cork bobs up and down in a lake. The distance from the bottom of the lake to the center of the cork at time $t \ge 0$ is given by $s(t) = 12 + \cos \pi t$, where $s(t)$ is in feet and t is in seconds.

(a) Describe the motion of the cork for $0 \le t \le 2$.

(b) During what time intervals is the cork rising?

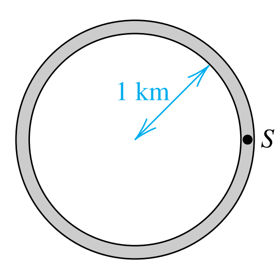
CHAPTER 6 DISCUSSION EXERCISES

1 Determine the number of solutions of the equation

$$\cos x + \cos 2x + \cos 3x = \pi.$$

2 **Racetrack coordinates** Shown in the figure is a circular racetrack of diameter 2 kilometers. All races begin at S and proceed in a counterclockwise direction. Approximate, to four decimal places, the coordinates of the point at which the following races end relative to a rectangular coordinate system with origin at the center of the track and S on the positive x-axis.

Exercise 2

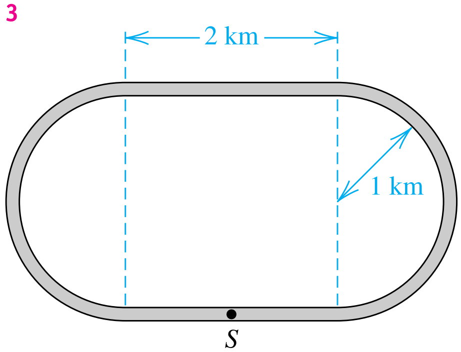


(a) A drag race of length 2 kilometers

(b) An endurance race of length 500 kilometers

3 Racetrack coordinates Work Exercise 2 for the track shown in the figure, if the origin of the rectangular coordinate system is at the center of the track and S is on the negative y-axis.

Exercise 3



4 Outboard motor propeller A 90-horsepower outboard motor at full throttle will rotate its propeller at 5000 revolutions per minute.

(a) Find the angular speed ω of the propeller in radians per second.

(b) The center of a 10-inch-diameter propeller is located 18 inches below the surface of the water. Express the depth $D(t) = a \cos(\omega t + c) + d$ of a point on the edge of a propeller blade as a function of time t, where t is in seconds. Assume that the point is initially at a depth of 23 inches.

5 Discuss the relationships among periodic functions, one-to-one functions, and inverse functions. With these relationships in mind, discuss what must happen for the trigonometric functions to have inverses.