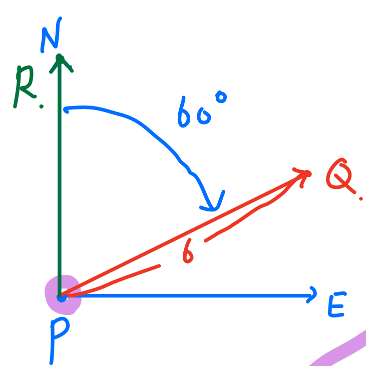
# Section 8.3 Continued.

**Ex** Two forces $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of both magnitudes 6.0 kilograms act at a point P. The direction of $\overrightarrow{PQ}$ is N60°E and the direction of $\overrightarrow{PR}$ is N.

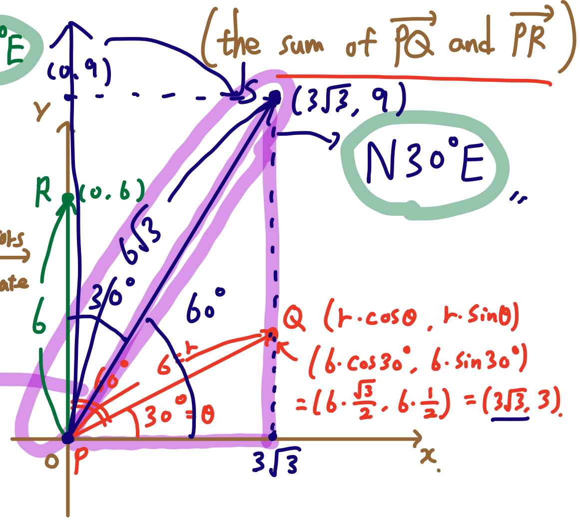
Find the magnitude and direction of the resultant $\overrightarrow{PS}$.

= $6\sqrt{3}$     = N30°E

(the sum of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$)

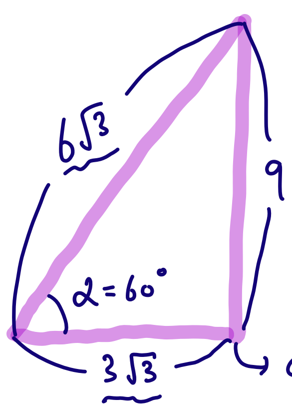



Put the vectors on the coordinate plane!




N30°E

$\overrightarrow{PR} = \langle 0, 6 \rangle$.    $\overrightarrow{PQ} = \langle 3\sqrt{3}, 3 \rangle$




$\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{PQ} = \langle 0, 6 \rangle + \langle 3\sqrt{3}, 3 \rangle = \langle 3\sqrt{3}, 9 \rangle$.

$\|\overrightarrow{PS}\| = \|\langle 3\sqrt{3}, 9 \rangle\| = \sqrt{(3\sqrt{3})^2 + 9^2} = \sqrt{9 \cdot 3 + 81}$

$= \sqrt{27 + 81}$

$= \sqrt{108}$

$= \sqrt{36 \cdot 3} = 6\sqrt{3}$

$\cos\alpha = \dfrac{3\sqrt{3}}{6\sqrt{3}} = \dfrac{1}{2}$.

$\Rightarrow \cos\alpha = \dfrac{1}{2}$, $\alpha = 60°$.

# Section 8.4 The Dot Product

∗ Product of real numbers is real number, but dot product of vectors is not a vector but a real number!

We know how to add two vectors $\mathbf{a}$ and $\mathbf{b}$.

Q : Can we multiply two vectors?

A : Yes!

There are two different products ($\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$), but we will only study the former one.

## Definition of the Dot Product.

Let $\mathbf{a} = \langle a_1, a_2 \rangle = a_1 i + a_2 j$ and $\mathbf{b} = \langle b_1, b_2 \rangle = b_1 i + b_2 j$.

The dot product of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \cdot \mathbf{b}$, is

vectors ← a real number.

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

∗ $\mathbf{a} \cdot \mathbf{b}$ : a dot b.

∗ "Dot product" is also called "Scalar product" or "inner product".

**Ex** Find $\mathbf{a} \cdot \mathbf{b}$.

✓ (a) $\mathbf{a} = \langle -4, 2 \rangle$, $\mathbf{b} = \langle 3, 5 \rangle$. $\langle -4, 2 \rangle \cdot \langle 3, 5 \rangle = (-4) \cdot 3 + 2 \cdot 5$
$= -12 + 10 = \boxed{-2}$

✓ (b) $\mathbf{a} = 3i - 4j$, $\mathbf{b} = 2i + 5j$. $\langle 3, -4 \rangle \cdot \langle 2, 5 \rangle = 3 \cdot 2 + (-4) \cdot 5$
$= \langle 3, -4 \rangle$ $= \langle 2, 5 \rangle$
$= 6 + (-20)$
$= \boxed{-14}$

## Properties of the Dot Product

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors and $m$ is a real number, then

✓ (1) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

$\mathbf{a} = \langle a_1, a_2 \rangle$.
$\mathbf{a} \cdot \mathbf{a} = \langle a_1, a_2 \rangle \cdot \langle a_1, a_2 \rangle = a_1^2 + a_2^2 = \left(\sqrt{a_1^2 + a_2^2}\right)^2 = \|\mathbf{a}\|^2$

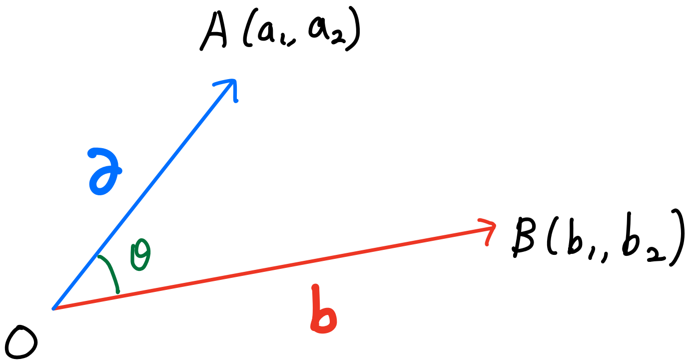
✓ (2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

✓ (3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(4) $(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$

✓ (5) $\mathbf{0} \cdot \mathbf{a} = 0$

$\langle 0, 0 \rangle \cdot \langle a_1, a_2 \rangle = 0 \cdot a_1 + 0 \cdot a_2 = 0 + 0 = 0$



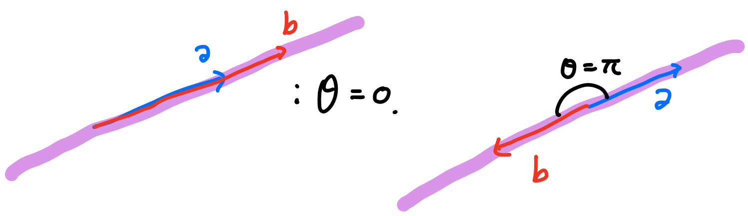

## Definition of Parallel and Orthogonal Vectors.

Let $\theta$ be the angle between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$.

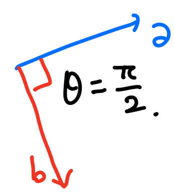
(1) $\mathbf{a}$ and $\mathbf{b}$ are "parallel" if $\theta = 0$ or $\theta = \pi$.

(2) $\mathbf{a}$ and $\mathbf{b}$ are "orthogonal" if $\theta = \frac{\pi}{2}$.

Ex (1) Parallel vectors




(2) Orthogonal vectors.



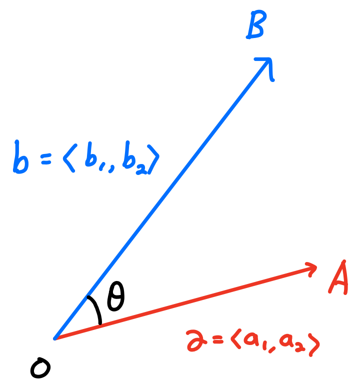

* We assume the zero vector $\mathbf{0}$ is parallel and orthogonal to every vector $\mathbf{a}$.

The following theorem relates the dot product and the angle determined by two vectors.

**Theorem on the Dot Product.**
If $\theta$ is the angle between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\boxed{\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos\theta}$$

<u>Proof</u> Using the law of Cosine to the triangle $\triangle OAB$:




* We will omit the proof of it.

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\cdot\|\mathbf{b}\|\cdot\cos\theta$$

$$\Big\downarrow \text{ divide by } \|\mathbf{a}\|\cdot\|\mathbf{b}\|$$

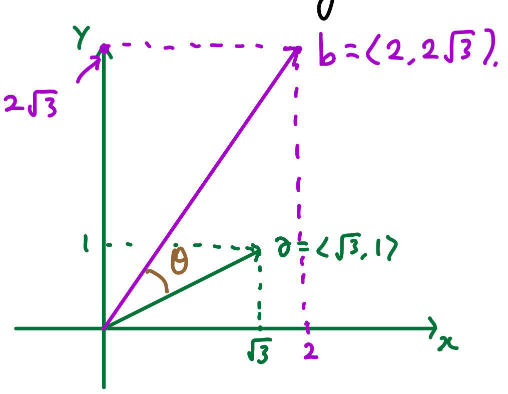
$$\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\cdot\|\mathbf{b}\|} = \cos\theta .$$

**Theorem on the Cosine of the Angle Between Vectors.**

If $\theta$ is the angle between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\cdot\|\mathbf{b}\|} .$$

Ex Find the angle between $\mathbf{a} = \langle \sqrt{3}, 1\rangle$ and $\mathbf{b} = \langle 2, 2\sqrt{3}\rangle$.




$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\cdot\|\mathbf{b}\|} = \frac{\cancel{4}^{1}\sqrt{3}}{2\cdot\cancel{4}_{1}} = \frac{\sqrt{3}}{2}.$$

$\mathbf{a} = \langle \sqrt{3}, 1\rangle \Rightarrow \|\mathbf{a}\| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2.$

$\mathbf{b} = \langle 2, 2\sqrt{3}\rangle \Rightarrow \|\mathbf{b}\| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4.$

$\mathbf{a}\cdot\mathbf{b} = \langle \sqrt{3}, 1\rangle \cdot \langle 2, 2\sqrt{3}\rangle = \sqrt{3}\cdot 2 + 1\cdot 2\sqrt{3}$

$= 2\sqrt{3} + 2\sqrt{3} = 4\sqrt{3}.$

$\cos\theta = \frac{\sqrt{3}}{2}. \Rightarrow \boxed{\theta = 30^\circ}$

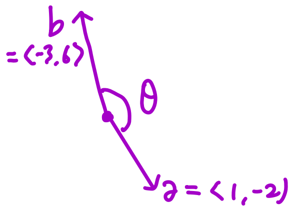
$= \langle 1, -2 \rangle$ $\quad\quad$ $= \langle -3, 6 \rangle.$

**Ex** Let $\mathbf{a} = 1i - 2j$ and $\mathbf{b} = -3i + 6j$.

(a) Show that $\mathbf{a}$ and $\mathbf{b}$ are parallel.

(b) Find the scalar $m$ such that $\mathbf{b} = m\mathbf{a}$.

(a)




$$\cos\theta = \frac{a \cdot b}{\|a\| \cdot \|b\|} = \frac{-15}{\sqrt{5} \cdot 3\sqrt{5}} = \frac{-15}{15} = -1.$$

$$a \cdot b = \langle 1, -2 \rangle \cdot \langle -3, 6 \rangle = 1 \cdot (-3) + (-2) \cdot 6$$
$$= -3 + (-12)$$
$$= -15.$$

$$\|a\| = \|\langle 1, -2 \rangle\| = \sqrt{1^2 + (-2)^2} = \sqrt{1 + 4} = \sqrt{5}$$

$$\|b\| = \|\langle -3, 6 \rangle\| = \sqrt{(-3)^2 + 6^2} = \sqrt{9 + 36} = \sqrt{45}$$
$$= \sqrt{9 \cdot 5}$$
$$= 3\sqrt{5}$$

$\cos\theta = -1 \Rightarrow \theta = \pi \Rightarrow$ Parallel!

(b) $\boxed{b = ma}$

$$-3i + 6j = m \cdot (i - 2j)$$

$$-3i + 6j = mi - 2mj$$

$\boxed{m = -3}$. $\quad -2m = 6$.

$\quad -2 \cdot (-3) = 6.$ ✓

Hence $\boxed{m = -3}$!