Seok Hyun Byun.

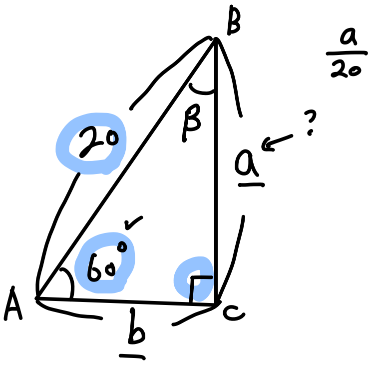
Section 8.1 The Law of Sines.

Recall : Solve the triangle :

$\beta : 60+90+\beta = 180. \rightarrow \beta = 30^\circ.$

$a : \sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{a}{20} \rightarrow a = 10\sqrt{3}$

$b : \cos 60^\circ = \frac{1}{2} = \frac{b}{20} \rightarrow b = 10$



Recall : Express a in terms of α and c :

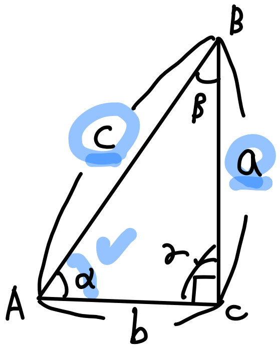
$\sin\alpha = \frac{a}{c}$

$\downarrow \times c$

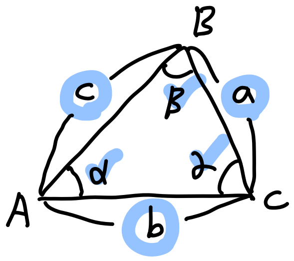
$$\boxed{a = c\cdot\sin\alpha.} \xrightarrow{\div\sin\alpha} \frac{a}{\sin\alpha} = c = \frac{c}{1} = \frac{c}{\sin 90^\circ}$$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin 90^\circ}$$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}, \text{ where } \gamma = 90^\circ.$$



In fact, more general is true!



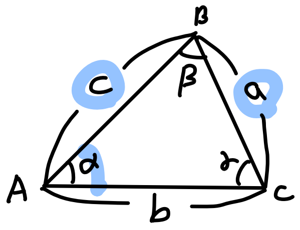
The Law of Sines

For any triangle △ABC, we have $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

Using this, we can solve several triangles.

* Caution : The triangles that satisfy the given condition may not be unique!

Case 1) Two sides and an angle opposite one of them are given.



Ex Solve $\triangle ABC$, given $\alpha = 30°$, $a = 2$, and $c = 2\sqrt{3}$

Recall : $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \longrightarrow \frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$.

$\longrightarrow \frac{2}{\sin 30°} = \frac{2\sqrt{3}}{\sin\gamma}$.

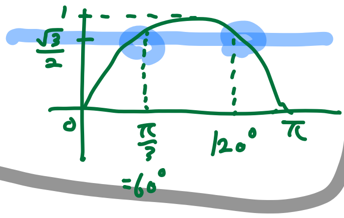
$\longrightarrow \frac{2}{\frac{1}{2}} = \frac{2\sqrt{3}}{\sin\gamma}$

$\left(\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc} \right)$

$\longrightarrow \frac{\frac{2}{1}}{\frac{1}{2}} = \frac{2 \cdot 2}{1 \cdot 1} = \boxed{\frac{4}{1} = \frac{2\sqrt{3}}{\sin\gamma}}$

$\longrightarrow \frac{1}{4} = \frac{\sin\gamma}{2\sqrt{3}}, \quad \sin\gamma = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

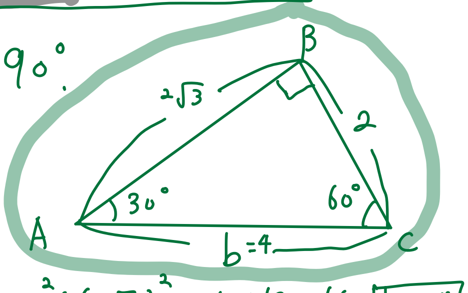
$\Rightarrow \sin\gamma = \frac{\sqrt{3}}{2}. \Rightarrow \gamma = 60° \text{ or } 120°.$



1) $\gamma = 60°$.

$+ \alpha = 30°, \ a = 2, \ c = 2\sqrt{3}$

$\Rightarrow \beta = 90°$.

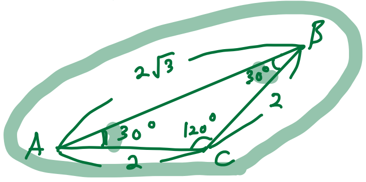


$\Rightarrow b^2 = 2^2 + (2\sqrt{3})^2 = 4 + 12 = 16. \ \boxed{b = 4}$

* There are two triangles that satisfy the given conditions!

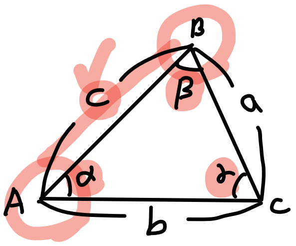
2) $\gamma = 120°, \quad \alpha = 30°, \ a = 2, \ c = 2\sqrt{3}$

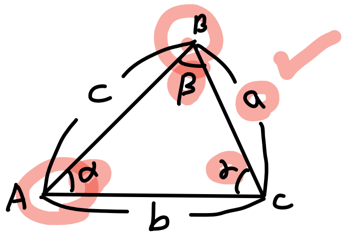
$\underset{30}{\cancel{\alpha}} + \beta + \underset{120}{\gamma} = 180 \Rightarrow \beta = 30°.$: equilateral triangle!



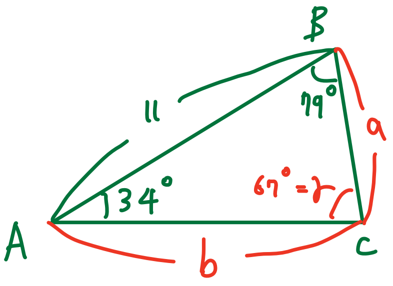
imply the third angle!

Case 2) Two angles and any side are given :





Ex Solve the triangle △ABC, given $\alpha = 34^\circ$, $\beta = 79^\circ$, and $c = 11$.



γ : $\underbrace{34 + 79}_{113} + \gamma = 180$, $\gamma = 67^\circ$.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow \frac{a}{\sin 34^\circ} = \boxed{\frac{b}{\sin 79^\circ} = \frac{11}{\sin 67^\circ}}.$$

$$\Rightarrow \frac{a}{\sin 34^\circ} = \frac{11}{\sin 67^\circ} \xrightarrow{\times \sin 34^\circ} \boxed{a = \frac{11 \cdot \sin 34^\circ}{\sin 67^\circ}}.$$

$$\Rightarrow \frac{b}{\sin 79^\circ} = \frac{11}{\sin 67^\circ} \xrightarrow{\times \sin 79^\circ} \boxed{b = \frac{11 \cdot \sin 79^\circ}{\sin 67^\circ}}$$

$\frac{\cancel{\sin}\, a}{\cancel{\sin}\, b} \overset{?}{=} \frac{a}{b}$: NOT TRUE

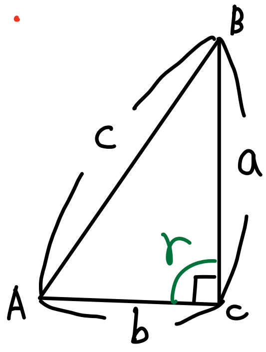
Example

$$2 = \frac{1}{\frac{1}{2}} = \frac{\sin 90^\circ}{\sin 30^\circ} = \frac{90^\circ}{30^\circ} = 3$$

they are different!

Section 8.2 The Law of Cosines.

Recall :



Pythagorean theorem :

$$c^2 = a^2 + b^2$$

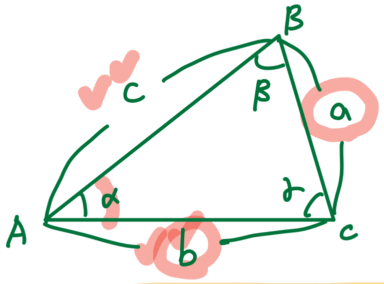
$$c^2 = a^2 + b^2 - 0$$

$$c^2 = a^2 + b^2 - 2ab \cdot 0$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos 90^\circ$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos \gamma \text{ when } \gamma = 90^\circ.$$

In fact, more general is true!



The Law of Cosines

For any triangle △ABC, we have

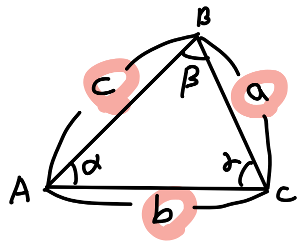
$$a^2 = \underline{b^2 + c^2} - 2bc\cos\alpha$$

$$b^2 = \underline{a^2 + c^2} - 2ac\cos\beta$$

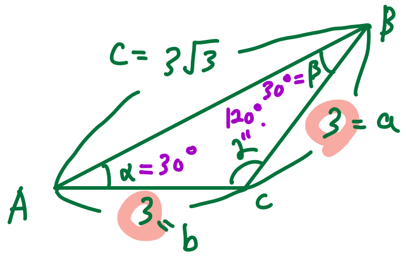
$$c^2 = \underline{a^2 + b^2} - 2ab\cos\gamma$$

Again, using this, we can solve some triangles.

Case 3) Three sides are given.



Ex Solve $\triangle ABC$, given $a = b = 3$ and $c = 3\sqrt{3}$.



$$a^2 = b^2 + c^2 - 2bc \cdot \cos\alpha$$

$$3^2 = 3^2 + (3\sqrt{3})^2 - \overset{6}{2 \cdot 3} \cdot 3\sqrt{3} \cdot \cos\alpha$$

$$9 = 9 + 27 - 18\sqrt{3} \cdot \cos\alpha$$

$$9 = 36 - 18\sqrt{3} \cdot \cos\alpha$$

$$-27 = -18\sqrt{3} \cdot \cos\alpha$$

$$\Big\downarrow \div(-18\sqrt{3})$$

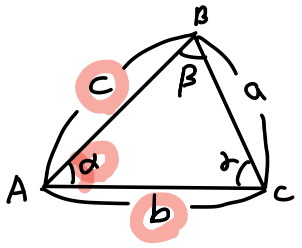
$$\cos\alpha = \frac{-27}{-18\sqrt{3}} = \frac{27}{18\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \boxed{\alpha = 30^\circ}$$

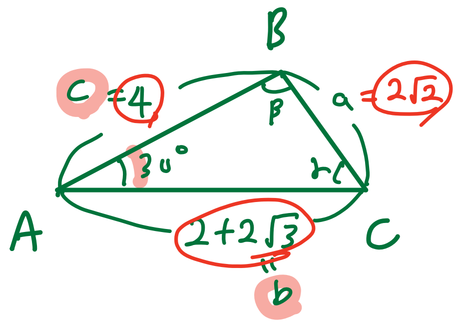
$$a = b \Rightarrow \alpha = \beta, \text{ so } \boxed{\beta = 30^\circ}$$

$$\boxed{\gamma = 180 - \alpha - \beta = 180 - 30 - 30 = 120^\circ}$$

Case 4) Two sides and the angle between them are given.



Ex Solve $\triangle ABC$, given $b = 2+2\sqrt{3}$, $c = 4$, and $\alpha = 30°$.



$$a^2 = b^2 + c^2 - 2b \cdot c \cdot \cos\alpha$$

$$a^2 = (2+2\sqrt{3})^2 + 4^2 - \cancel{2} \cdot (2+2\sqrt{3}) \cdot 4 \cdot \frac{\sqrt{3}}{\cancel{2}}.$$

$$= 4+12+8\sqrt{3} + 16 - 4(2+2\sqrt{3}) \cdot \sqrt{3}$$

$$= 16 + 8\sqrt{3} + 16 - (8+8\sqrt{3})\sqrt{3}$$

$$= 32 + 8\sqrt{3} - (8\sqrt{3} + 24)$$

$$= 32 + \cancel{8\sqrt{3}} - \cancel{8\sqrt{3}} - 24$$

$$= 8.$$

$$\Rightarrow \boxed{a = \sqrt{8} = 2\sqrt{2}}$$

Now we know all side lengths, so the problem reduces to Case 3)!