HW 11 : due 12/3 (Friday) at 11:59 pm

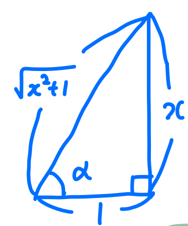
Special Written HW : due 12/12 (Sunday) at 11:59 pm

# Section 7.6 Continued

**Ex** Write the expression as an algebraic expression in $x$ for $x>0$

: $\sin(\tan^{-1} x) = \sin\alpha = \dfrac{x}{\sqrt{x^2+1}}$.

Let $\alpha = \tan^{-1} x$. $\longleftrightarrow$ $x = \tan\alpha = \dfrac{x}{1}$.

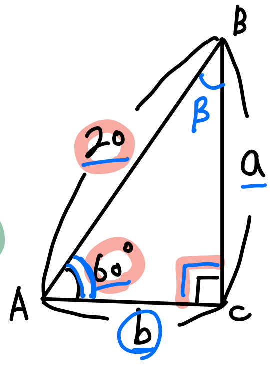



$\rightarrow$ $\sin\alpha = \dfrac{x}{\sqrt{x^2+1}}$.

Hence, $\sin(\tan^{-1} x) = \sin\alpha = \dfrac{x}{\sqrt{x^2+1}}$

# Section 8.1 The Law of Sines.

Recall: Solve the triangle:



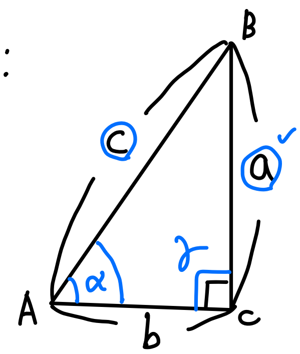

$\beta$ : $60+90+\beta=180.$ $\beta=30°$.

$a$ : $\sin 60° = \frac{\sqrt{3}}{2} = \frac{a}{20} \longrightarrow a = 10\sqrt{3}.$

$b$ : $\cos 60° = \frac{1}{2} = \frac{b}{20} \longrightarrow b = 10.$

$$\frac{\sin 60° = \frac{a}{20}}{\cos 60° = \frac{b}{20}}$$

Recall: Express a in terms of α and c:




$$\sin\alpha = \frac{a}{c}$$

$$\Big\downarrow \times c$$

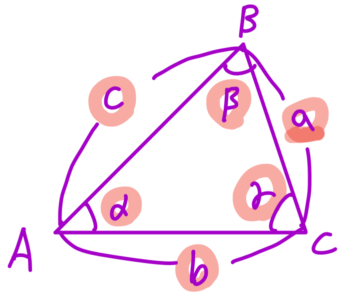
$$\boxed{a = c\sin\alpha} \xrightarrow{\div \sin\alpha} \frac{a}{\sin\alpha} = c$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{c}{1}$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{c}{\sin 90°}$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \quad \text{where } \gamma = 90°.$$

In fact, more general is true!



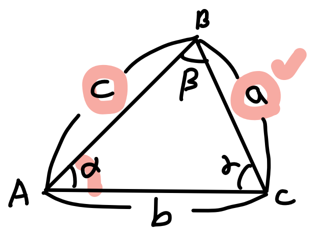

## The Law of Sines

For any triangle $\triangle ABC$, we have $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

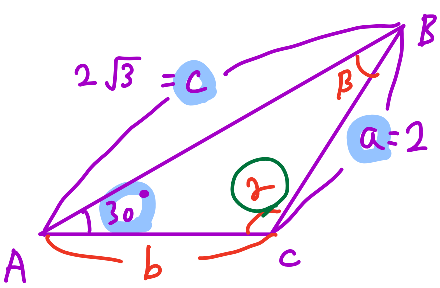
Using this, we can solve several triangles.

* Caution : The triangles that satisfy the given condition may not be unique!

## Case 1) Two sides and an angle opposite one of them are given.




**Ex** Solve $\triangle ABC$, given $\alpha = 30°$, $a = 2$, and $c = 2\sqrt{3}$




$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

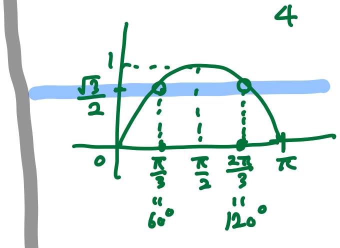
$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \; : \; \frac{2}{\sin 30°} = \frac{2\sqrt{3}}{\sin\gamma}.$$

$$\frac{2}{\frac{1}{2}} = \frac{2\sqrt{3}}{\sin\gamma}.$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

$$\frac{4}{1} = \frac{2\sqrt{3}}{\sin\gamma}.$$

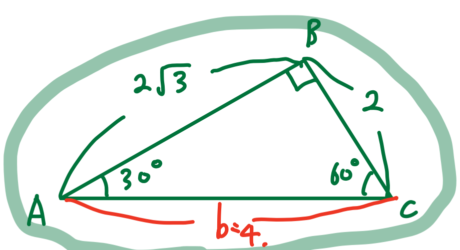
$$\frac{1}{4} = \frac{\sin\gamma}{2\sqrt{3}}, \quad \sin\gamma = \frac{2\sqrt{3}}{4} = \boxed{\frac{\sqrt{3}}{2}}$$




$\Rightarrow \gamma = 60°, 120°$

(1) $\gamma = 60°$.
$\alpha = 30°$, $a = 2$, $c = 2\sqrt{3}$

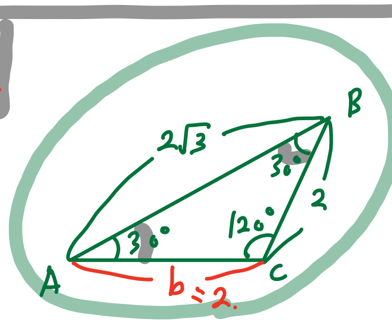
$\alpha + \beta + \gamma = 180°$
$30° + \beta + 60° = 180° \Rightarrow \beta = 90°$.




$b^2 = 2^2 + (2\sqrt{3})^2 = 4 + 12 = 16$. $b^2 = 16$, $b = 4$.

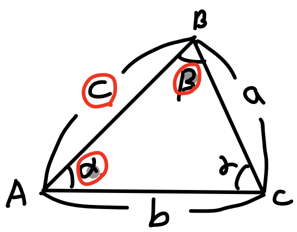
(2) $\gamma = 120°$, $\alpha = 30°$, $a = 2$, $c = 2\sqrt{3}$

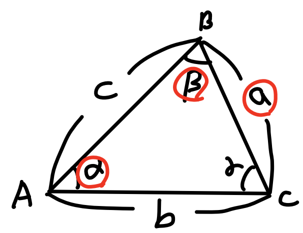
$30 + \beta + 120 = 180$, $\beta = 30°$.



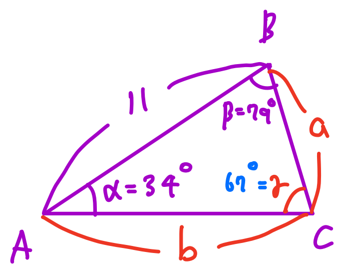
"the third angle"

## Case 2) Two angles and any side are given:




**Ex** Solve the triangle $\triangle ABC$, given $\alpha = 34°$, $\beta = 79°$, and $c = 11$.




$$\alpha + \beta + \gamma = 180°$$

$$34° + 79° + \gamma = 180°$$

$$113° + \gamma = 180°, \quad \gamma = 67°.$$

Recall: $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \Rightarrow \frac{a}{\sin 34°} = \frac{11}{\sin 67°}$$

$$\Rightarrow a = \frac{11 \sin 34°}{\sin 67°}$$

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow \frac{b}{\sin 79°} = \frac{11}{\sin 67°}$$

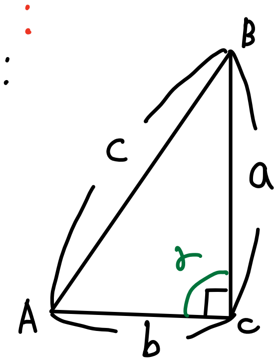
$$\Rightarrow b = \frac{11 \cdot \sin 79°}{\sin 67°}$$

$$\frac{\cancel{\sin} 79°}{\cancel{\sin} 67°} \neq \frac{79°}{67°}$$

∴ We cannot cancel out sin!

# Section 8.2 The Law of Cosines.

Recall :




Pythagorean theorem :

$c^2 = a^2 + b^2$

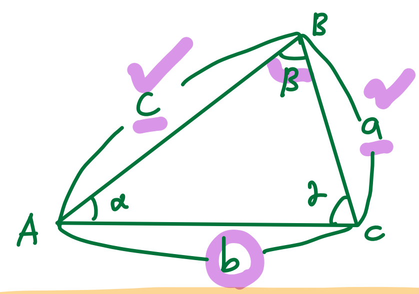
$c^2 = a^2 + b^2 - 0$

$c^2 = a^2 + b^2 - \underline{\underline{2ab \cdot 0}}$

$c^2 = a^2 + b^2 - 2ab \cdot \underbrace{\cos 90^\circ}_{0}$.

$c^2 = a^2 + b^2 - 2ab \cdot \cos\gamma$ where $\gamma = 90^\circ$.

In fact, more general is true!




## The Law of Cosines

For any triangle $\triangle ABC$, we have

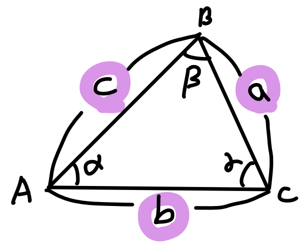
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

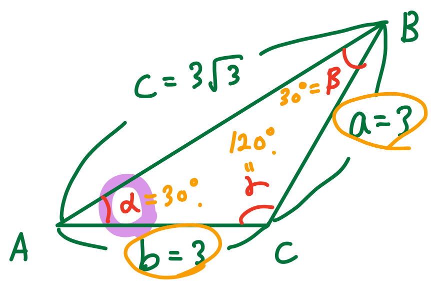
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Again, using this, we can solve some triangles.

Case 3) Three sides are given.




Ex Solve $\triangle ABC$, given $a = b = 3$ and $c = 3\sqrt{3}$.




$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$3^2 = 3^2 + (3\sqrt{3})^2 - 2 \cdot 3 \cdot 3\sqrt{3} \cdot \cos\alpha.$$

$$9 = 9 + 27 - 18\sqrt{3} \cdot \cos\alpha.$$

$$9 = \cancel{36} - 18\sqrt{3} \cdot \cos\alpha.$$
$$-36 \qquad -\cancel{36}.$$

$$-27 = -18\sqrt{3} \cdot \cos\alpha.$$

$$\cos\alpha = \frac{-27}{-18\sqrt{3}} = \frac{27}{18\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}.$$

$$\cos\alpha = \frac{\sqrt{3}}{2}, \quad \alpha = 30°.$$

Since $a = b$, $\alpha = \beta = 30°$.

$$30° + 30° + \gamma = 180° \Rightarrow \gamma = 120°$$