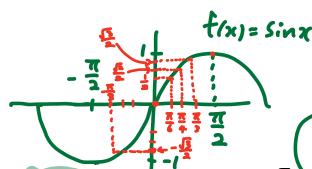
Section 7.6 Continued

cos and $\cos^{-1}$ also satisfy the following relations!

$\cos(\cos^{-1}x) = x$ for all x in the domain of $\cos^{-1}$ = $[-1, 1]$.

$\cos^{-1}(\cos y) = y$ for all y in the domain of cos = $[0, \pi]$.
(restricted)



Ex Find the exact value:

(a) $\cos\left[\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right] = \boxed{-\frac{\sqrt{2}}{2}}$ (b) $\cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \boxed{\frac{5\pi}{6}}$. (c) $\cos^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right] = \boxed{\frac{5\pi}{6}}$

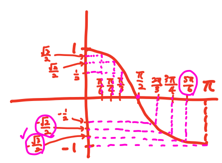
$\sin\left(-\frac{\pi}{3}\right) \to -\frac{\sqrt{3}}{2}$

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \alpha = \frac{3\pi}{4}$.

$\cos\alpha = -\frac{\sqrt{3}}{2}$ and α is in $[0, \pi]$.

$\Rightarrow \alpha = \frac{3\pi}{4}$.

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta = \frac{5\pi}{6}$



$\cos\theta = -\frac{\sqrt{3}}{2}$ and θ is in $[0, \pi]$

$\hookrightarrow \theta = \frac{5\pi}{6}$.

Ex Find the exact value of $\sin\left[\arccos\left(-\frac{1}{3}\right)\right]$.

$\sin\left(\cos^{-1}\left(-\frac{1}{3}\right)\right) = \sin\alpha = \boxed{\frac{2\sqrt{2}}{3}}$

Let us give a name for $\cos^{-1}\left(-\frac{1}{3}\right)$, α.

$\boxed{\alpha = \cos^{-1}\left(-\frac{1}{3}\right)}$. $\longleftrightarrow$ $\cos\alpha = -\frac{1}{3}$.

it is in $[0, \pi]$

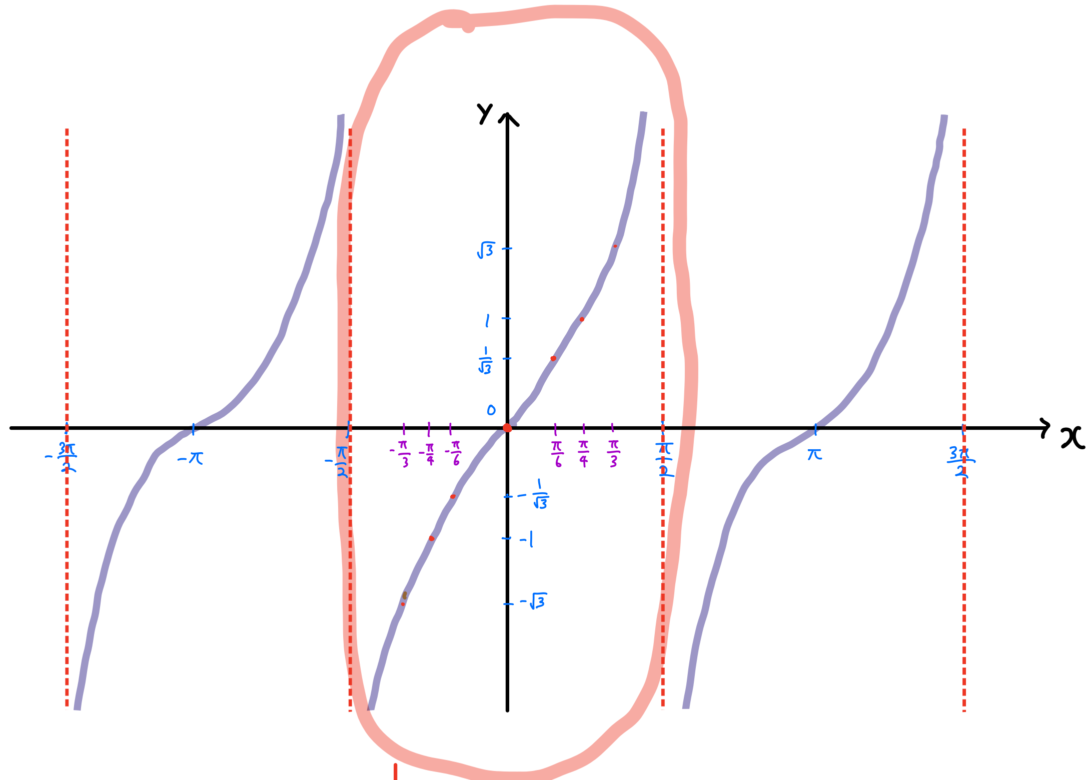
$\sin^2\alpha + \cos^2\alpha = 1$

$\boxed{\text{Recall: } \sin^2 x + \cos^2 x = 1.}$

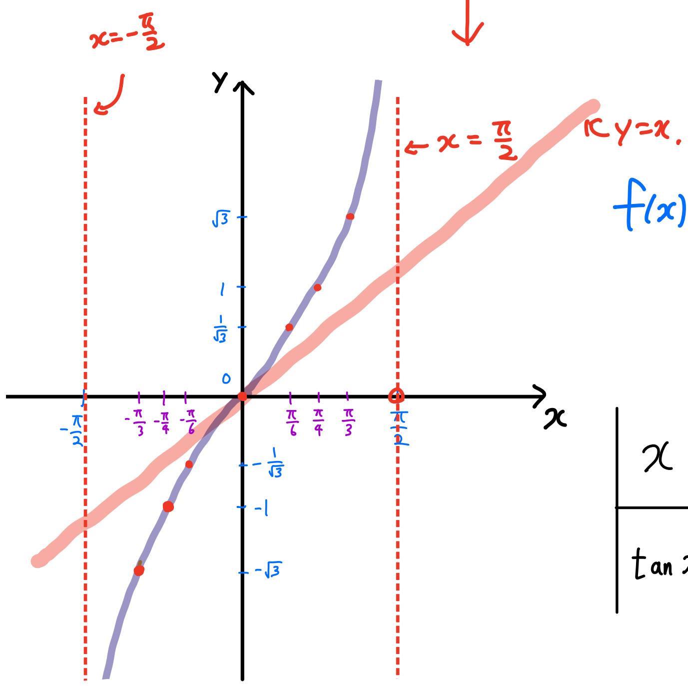
$\sin^2\alpha + \left(-\frac{1}{3}\right)^2 = 1$, $\sin^2\alpha + \frac{1}{9} = 1$. $\sin^2\alpha = \frac{8}{9}$. $\sin\alpha = \pm\sqrt{\frac{8}{9}} = \pm\frac{\sqrt{8}}{\sqrt{9}} = \pm\frac{2\sqrt{2}}{3}$

Hence, $\boxed{\sin\alpha = \frac{2\sqrt{2}}{3}}$ or $\cancel{-\frac{2\sqrt{2}}{3}}$ (because sin is nonnegative on $[0, \pi]$)

How about tangent ?



Restrict the domain of tangent to $(-\frac{\pi}{2}, \frac{\pi}{2})$



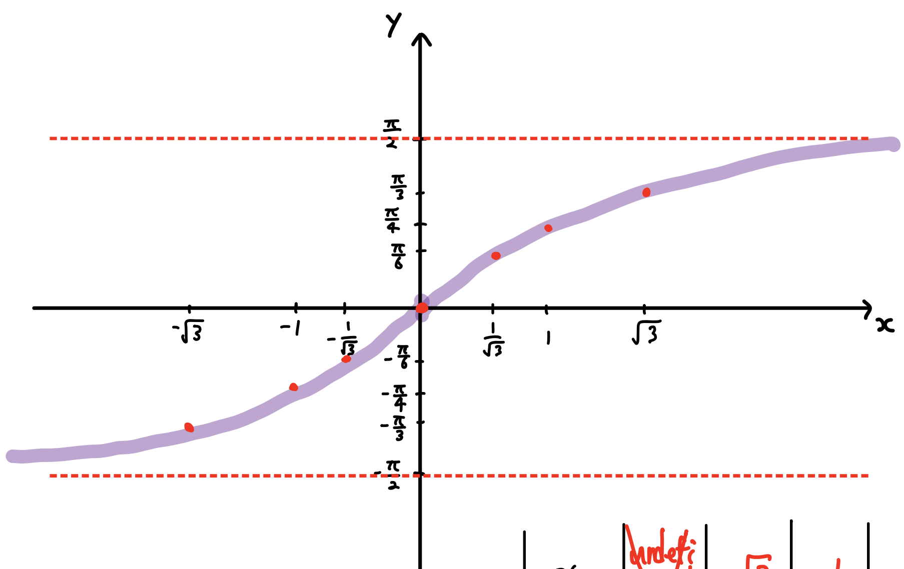
$f(x) = \tan x$ with $\begin{cases} \text{domain} = (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \text{range} = (-\infty, \infty) \end{cases}$

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined.

The inverse tangent function (arctangent function), denoted by $\tan^{-1}$ (arctan), is defined by

$$y = \tan^{-1} x \ (y = \arctan x) \quad \text{if and only if} \quad x = \tan y$$

for any real number x and for $-\frac{\pi}{2} < y < \frac{\pi}{2}$



$f^{-1}(x) = \tan^{-1} x$

- Domain $= (-\infty, \infty)$
- Range $= (-\frac{\pi}{2}, \frac{\pi}{2})$

x	~~undefined~~	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	~~undefined~~
$\tan^{-1} x$	$\cancel{-\frac{\pi}{2}}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\cancel{\frac{\pi}{2}}$

tan and $\tan^{-1}$ also satisfy the following relations!

$$\begin{cases} \tan(\tan^{-1} x) = x & \text{for all } x \text{ in the domain of } \tan^{-1} = (-\infty, \infty) \\ \tan^{-1}(\tan y) = y & \text{for all } y \text{ in the } \underbrace{\text{domain of } \tan}_{\text{(restricted)}} = (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$$

Ex Find the exact value:

(a) $\tan(\tan^{-1} 250)$

$= 250.$

(b) $\tan^{-1}\left(\tan \frac{\pi}{3}\right)$

$= \tan^{-1}(\sqrt{3})$

$= \frac{\pi}{3}.$

(c) $\tan^{-1}\left(\tan \frac{5\pi}{6}\right)$

$\tan \frac{5\pi}{6} = \tan\left(\frac{5\pi}{6} - \pi\right)$

$= \tan\left(-\frac{\pi}{6}\right).$

$\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}$

Ex Find the exact value of $\csc\left(\overbrace{\arctan \frac{1}{3}}^{\alpha}\right)$. $= \csc \alpha = \frac{1}{\sin \alpha}$

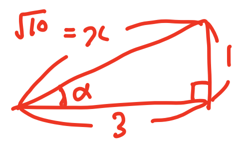
$= \sqrt{10}$

Let $\alpha = \arctan\left(\frac{1}{3}\right) \longleftrightarrow \tan \alpha = \frac{1}{3}.$

α is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$

α is in $\left(0, \frac{\pi}{2}\right)$

(This is because $\tan \alpha$ is positive)



$x^2 = 1^2 + 3^2 = 1 + 9 = 10.$ $x = \sqrt{10}$

Then, $\sin \alpha = \frac{1}{\sqrt{10}}$, and $\frac{1}{\sin \alpha} = \frac{1}{\frac{1}{\sqrt{10}}}$

$= \sqrt{10}.$

$= \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

Ex Find the exact value of $\sin\left(\underbrace{\arctan\frac{2}{3}}_{\alpha} - \underbrace{\arccos\frac{3}{4}}_{\beta}\right)$

Let $\alpha = \arctan\frac{2}{3}$ and $\beta = \arccos\frac{3}{4}$.

Then we have to find $\sin(\alpha - \beta)$.

By the subtraction formula, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

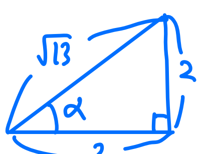
Hence, to find the value of $\sin(\alpha-\beta)$, we need to find the values of $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$.

$\alpha = \arctan\frac{2}{3}$. $\tan\alpha = \frac{2}{3}$

α is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

α is in $\left(0, \frac{\pi}{2}\right)$

(because $\tan\alpha$ is positive)



$\Rightarrow \sin\alpha = \frac{2}{\sqrt{13}}$, $\cos\alpha = \frac{3}{\sqrt{13}}$

$\beta = \arccos\frac{3}{4}$. $\cos\beta = \frac{3}{4}$.

β is in $[0, \pi]$

$\sin^2\beta + \cos^2\beta = 1$.

$\sin^2\beta + \frac{9}{16} = 1$, $\sin^2\beta = \frac{7}{16}$,

$\boxed{\sin\beta = \frac{\sqrt{7}}{4}}$ or ~~$-\frac{\sqrt{7}}{4}$~~

(because sin is positive on $[0,\pi]$)

Hence, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{2}{\sqrt{13}}\cdot\frac{3}{4} - \frac{3}{\sqrt{13}}\cdot\frac{\sqrt{7}}{4}$

$= \frac{6}{4\sqrt{13}} - \frac{3\sqrt{7}}{4\sqrt{13}} = \frac{6-3\sqrt{7}}{4\sqrt{13}}$