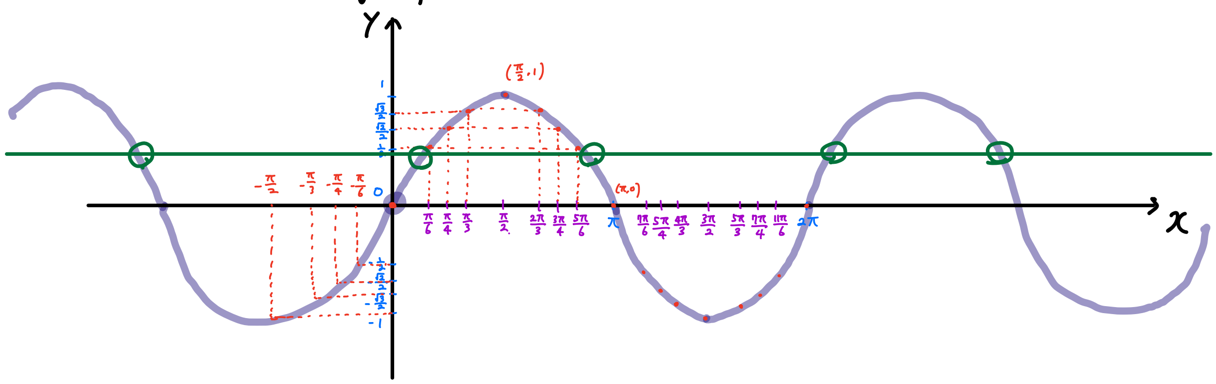
Section 7.6. The Inverse Trigonometric Functions.

Recall : The graph of $f(x) = \sin x$

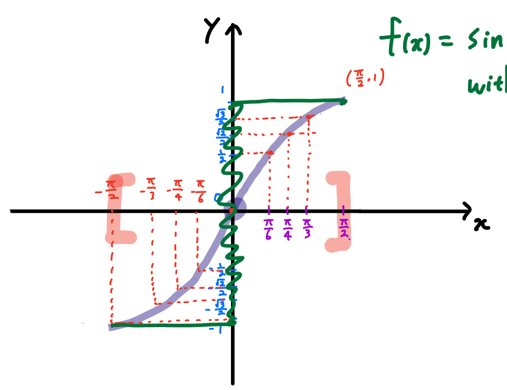


It is a graph of function, but not a graph of one-to-one function...

↳ horizontal line test!

Q. How can I have a graph of one-to-one function... ?

A. Restrict the domain! $(-\infty, \infty) \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$.



$f(x) = \sin x$
with domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

The function $f(x) = \sin x$ with domain: $[-\frac{\pi}{2}, \frac{\pi}{2}]$ & range: $[-1, 1]$ is one-to-one function, so we can consider its inverse function!

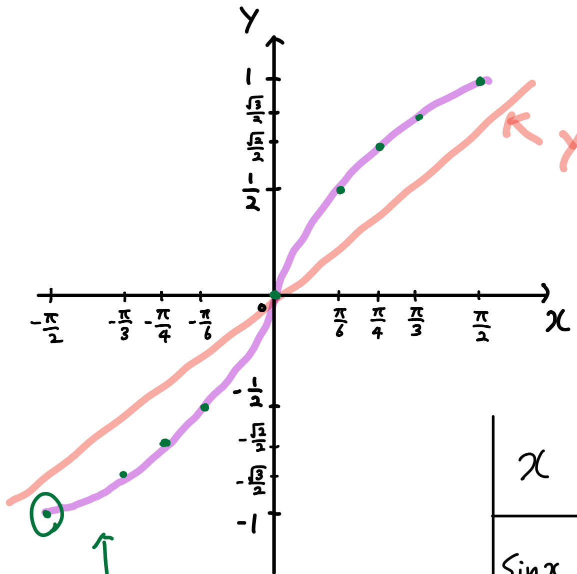
The inverse sine function, denoted by $\sin^{-1}$ ($\arcsin$), is defined by
(arcsine function.)

$$y = \sin^{-1} x \ (y = \arcsin x) \quad \text{if and only if} \quad x = \sin y$$

for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Ex) $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\Rightarrow \sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$.



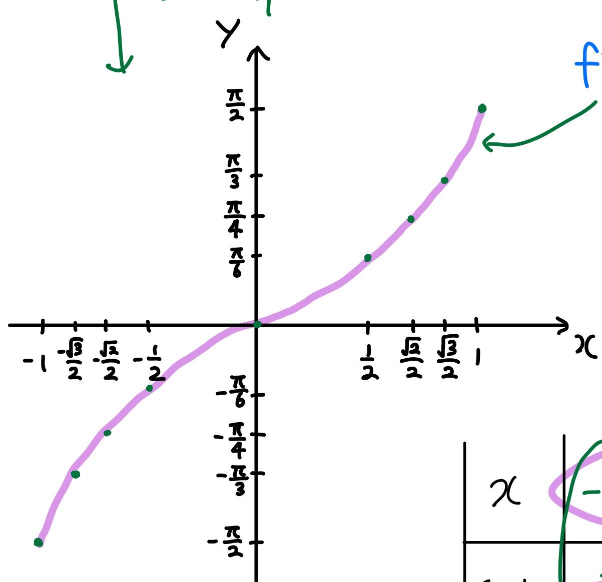
$f(x) = \sin x$ with domain $= [-\frac{\pi}{2}, \frac{\pi}{2}]$, range $= [-1, 1]$.

domain.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

range.

Graphs are symmetric with respect to $y = x$.



$f^{-1}(x) = \sin^{-1} x$: domain $= [-1, 1]$, range $= [-\frac{\pi}{2}, \frac{\pi}{2}]$.

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1} x$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

* Notation! $\sin^2 x = \sin x \cdot \sin x$

$\sin^1 x = \sin x$

$\sin^0 x = 1$

inverse sine function. $\sin^{-1} x = \cancel{\dfrac{1}{\sin x}}$, but $(\sin x)^{-1} = \dfrac{1}{\sin x}$.

$\sin^{-2} x = \dfrac{1}{\sin^2 x} = \dfrac{1}{\sin x \cdot \sin x}$

The same is true for cosine and tangent! $\cos^{-1} x$, $\tan^{-1} x$ are different from $\dfrac{1}{\cos x}$ and $\dfrac{1}{\tan x}$.

*Recall: For any one to one function f and its inverse function f^{-1}

$f(f^{-1}(x))$ ← $(f \circ f^{-1})(x) = x$ for all x in the domain of f^{-1}

$f^{-1}(f(y))$ ← $(f^{-1} \circ f)(y) = y$ for all y in the domain of f.

$f(x) = \sin x$ with domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $f^{-1}(x) = \sin^{-1} x$ are inverse to each other!

Hence, we have: $\sin(\sin^{-1} x) = x$ for all x in the domain of $\sin^{-1}$ $= [-1, 1]$.

$\sin^{-1}(\sin y) = y$ for all y in the domain of $\sin$ $= [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Ex Find the exact value:

(a) $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$

$\left(\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \Rightarrow \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}\right)$

$= \sin\frac{\pi}{3} = \boxed{\frac{\sqrt{3}}{2}}$

(b) $\sin^{-1}\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}.$

$= \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

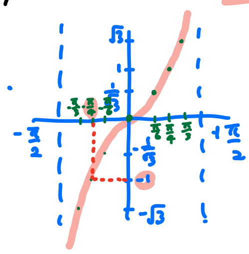
(c) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

$\sin\frac{5\pi}{6} = \frac{1}{2}.$

$= \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}.$

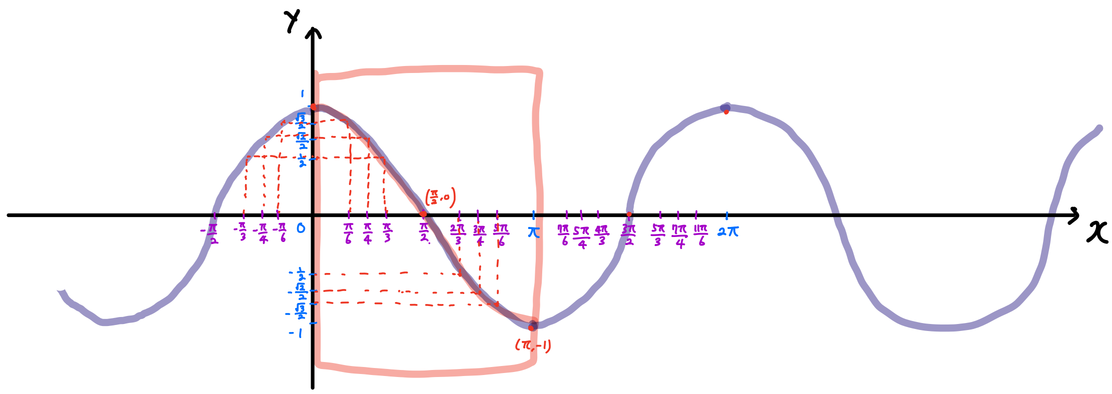
Ex Find the exact value of y if $y = \sin^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$

First, find $\tan\left(-\frac{\pi}{4}\right) = -1.$

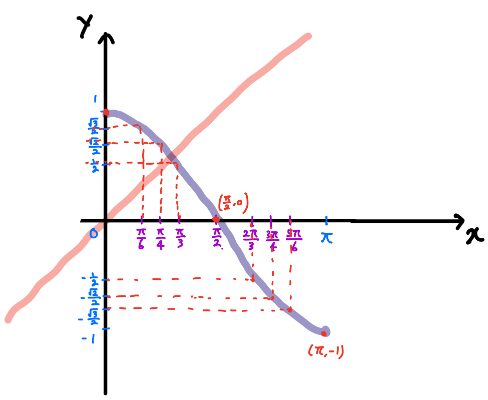


Thus, $y = \sin^{-1}\left(\underbrace{\tan\left(-\frac{\pi}{4}\right)}_{-1}\right) = \sin^{-1}(-1) = \boxed{-\frac{\pi}{2}}$

How about cosine ?



Restrict the domain of cosine : $(-\infty, \infty) \longrightarrow \boxed{[0, \pi]}$.



$f(x) = \cos x$ with domain $= [0, \pi]$,
range $= [-1, 1]$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

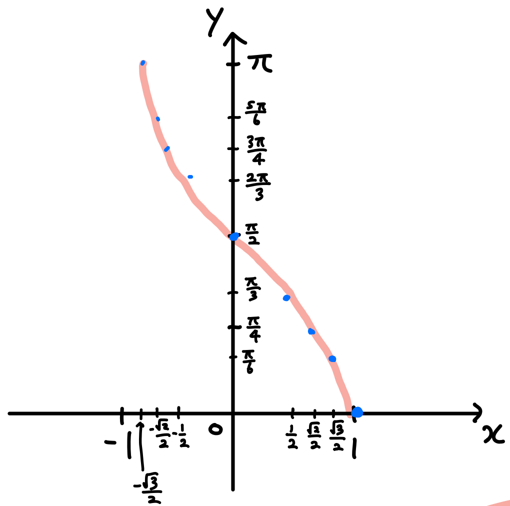
The inverse cosine function (arccosine function.), denoted by $\cos^{-1}$ (arccos), is defined by

$y = \cos^{-1} x$ ($y = \arccos x$) if and only if $x = \cos y$

for $-1 \le x \le 1$ and $0 \le y \le \pi$.

Ex. $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \rightarrow \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

$\arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$.



$f^{-1}(x) = \cos^{-1} x$: domain $= [-1, 1]$.

range $= [0, \pi]$.

x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cos^{-1}x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π