HW 10 : due next Monday at 11:59 pm.

Section 7.3 The Addition and Subtraction Formulas.

↳ $\cos(\alpha+\beta)$, $\sin(\alpha-\beta)$, ...

In this section, we will see several formulas.

The first formula is the following :

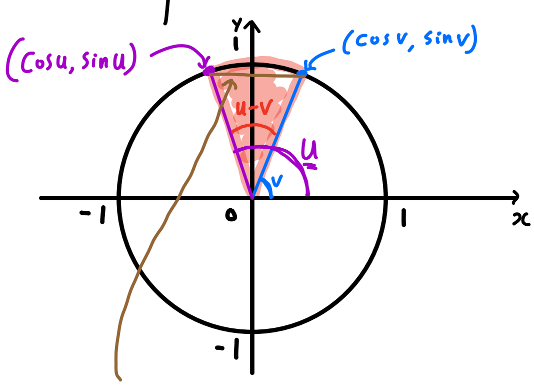
Subtraction Formula for cosine.

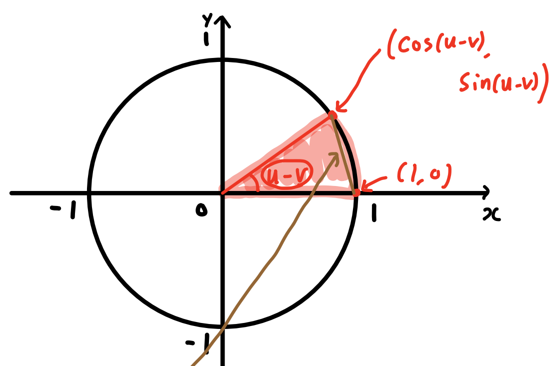
$$\boxed{\cos(\underset{45^\circ}{u} - \underset{30^\circ}{v}) = \cos u \cos v + \sin u \sin v}$$

Ex $\cos(\underbrace{15^\circ}_{45^\circ-30^\circ}) = \cos(45^\circ - 30^\circ)$

$= \cos 45^\circ \cdot \cos 30^\circ + \sin 45^\circ \cdot \sin 30^\circ$

How to prove it ?





distance between (cos u, sin u) and (cos v, sin v) = distance between (1, 0) and (cos(u-v), sin(u-v))

= (distance formula) $\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2}$ = $\sqrt{(1-\cos(u-v))^2 + (0 - \sin(u-v))^2}$

$(\cos u - \cos v)^2 + (\sin u - \sin v)^2 = (1 - \cos(u-v))^2 + (0 - \sin(u-v))^2$

$\cos^2 u - 2\cos u \cos v + \cos^2 v + \sin^2 u - 2 \sin u \sin v + \sin^2 v = 1 - 2\cos(u-v) + \cos^2(u-v) + \sin^2(u-v)$

$\cancel{2} - 2\cos u \cos v - 2 \sin u \sin v = \cancel{2} - 2\cos(u-v) \xrightarrow{\div 2} \boxed{\cos(u-v) = \cos u \cos v + \sin u \sin v.}$

Recall that $f(x) = \cos x$ is even function and $f(x) = \sin x$ is an odd function.

$\Downarrow$ $\cos(-x) = \cos x$. $\Downarrow$ $\sin(-x) = -\sin x$.

subtraction formula.

Then, $\cos(u+v) = \cos(u-(-v)) = \cos u \cdot \underbrace{\cos(-v)}_{\cos v} + \sin u \cdot \underbrace{\sin(-v)}_{-\sin v}$

$= \boxed{\cos u \cdot \cos v - \sin u \cdot \sin v.}$

Hence, we just obtain the addition formula for cosine:

Addition formula for cosine.

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

Ex Find $\cos 15°$ and $\cos \frac{5\pi}{12}$

$\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$.

$\cos 15°$

$= \cos(45° - 30°)$

$= \cos 45° \cdot \cos 30° + \sin 45° \cdot \sin 30°$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

(* You will get the same answer if you use $15° = 60° - 45°$.)

$\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

$= \cos\frac{\pi}{4} \cdot \cos\frac{\pi}{6} - \sin\frac{\pi}{4} \cdot \sin\frac{\pi}{6}$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

Using the subtraction formula for cosine, we can also prove the following "Cofunction Formulas".

Cofunction Formulas

If u is a real number or the radian measure of an angle, then

(1) $\cos\left(\frac{\pi}{2} - u\right) = \sin u$

(2) $\sin\left(\frac{\pi}{2} - u\right) = \cos u$

(3) $\tan\left(\frac{\pi}{2} - u\right) = \cot u$

(4) $\sec\left(\frac{\pi}{2} - u\right) = \csc u$

(5) $\csc\left(\frac{\pi}{2} - u\right) = \sec u$

(6) $\cot\left(\frac{\pi}{2} - u\right) = \tan u$

Proof) (1) $\cos\left(\frac{\pi}{2} - u\right) \overset{\text{subtraction formula}}{=} \underbrace{\cos\frac{\pi}{2}}_{=0}\cos u + \underbrace{\sin\frac{\pi}{2}}_{1}\sin u = 0 + \sin u = \sin u$

(2) We know $\cos\left(\frac{\pi}{2} - u\right) = \sin u$ (from (1))

Replace u by $\left(\frac{\pi}{2} - u\right)$ $\rightarrow$ $\cos\left(\underbrace{\frac{\pi}{2} - \left(\frac{\pi}{2} - u\right)}_{\cancel{\frac{\pi}{2}} - \cancel{\frac{\pi}{2}} + u}\right) = \sin\left(\frac{\pi}{2} - u\right)$

$\boxed{\cos u = \sin\left(\frac{\pi}{2} - u\right)}$

Using the subtraction formula for cosine and the cofunction formula,

now we can prove

"Addition and Subtraction Formulas for sine and tangent"

(1) $\sin(u+v) = \sin u \cos v + \cos u \sin v$

(2) $\sin(u-v) = \sin u \cos v - \cos u \sin v$

(3) $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

(4) $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

For example, if we want to find the value of $\tan 75°$,

by (3), $\tan 75° = \tan(45° + 30°) = \dfrac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$$

$$= \frac{\left(1 + \frac{1}{\sqrt{3}}\right)}{\left(1 - \frac{1}{\sqrt{3}}\right)} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Ex Suppose $\sin\alpha = \frac{3}{5}$ and $\cos\beta = -\frac{5}{13}$ where α is in quadrant I and β is in quadrant II.

Find the exact value of $\sin(\alpha+\beta)$

$\sin\alpha = \frac{3}{5}$, $\cos\alpha = ?$

Recall that $\sin^2\alpha + \cos^2\alpha = 1$

$$\left(\frac{3}{5}\right)^2 + \cos^2\alpha = 1$$

$$\frac{9}{25} + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \frac{9}{25} = \frac{25}{25} - \frac{9}{25} = \frac{16}{25} = \frac{4^2}{5^2}$$

$\boxed{\cos\alpha = \frac{4}{5}}$ or $\cancel{-\frac{4}{5}}$

← b/c cos is positive on the quadrant I.

$\cos\beta = -\frac{5}{13}$, $\sin\beta = ?$

Recall that $\sin^2\beta + \cos^2\beta = 1$

$$\sin^2\beta + \left(-\frac{5}{13}\right)^2 = 1$$

$$\sin^2\beta + \frac{25}{169} = 1$$

$$\sin^2\beta = 1 - \frac{25}{169} = \frac{169}{169} - \frac{25}{169} = \frac{144}{169} = \frac{12^2}{13^2}$$

$\boxed{\sin\beta = \frac{12}{13}}$ or $\cancel{-\frac{12}{13}}$

← b/c Sin is positive on the quadrant II.

$$\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta.$$

$$= \frac{3}{5}\cdot\left(-\frac{5}{13}\right) + \frac{4}{5}\cdot\left(\frac{12}{13}\right) = -\frac{15}{65} + \frac{48}{65} = \boxed{\frac{33}{65}}$$