(Section 6.2 Continued.)

Ex Let θ be an acute angle. Express $\tan\theta$ in terms of $\sin\theta$.

↳ all the trigonometric function's value are positive!

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$

$\cos\theta$ ← $\sqrt{1-\sin^2\theta}$

$\cos^2\theta = 1 - \sin^2\theta$.

$\cos\theta = \pm\sqrt{1-\sin^2\theta}$

↳ Since $\cos\theta$ is positive, $\cos\theta = \sqrt{1-\sin^2\theta}$.

$$\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

Ex Verify the following identity: $\cot\theta + \tan\theta = \csc\theta \cdot \sec\theta$

Recall $\sin\theta$, $\cos\theta$

$\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\cot\theta + \tan\theta = \frac{\cos\theta \times \cos\theta}{\sin\theta \times \cos\theta} + \frac{\sin\theta \times \sin\theta}{\cos\theta \times \sin\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

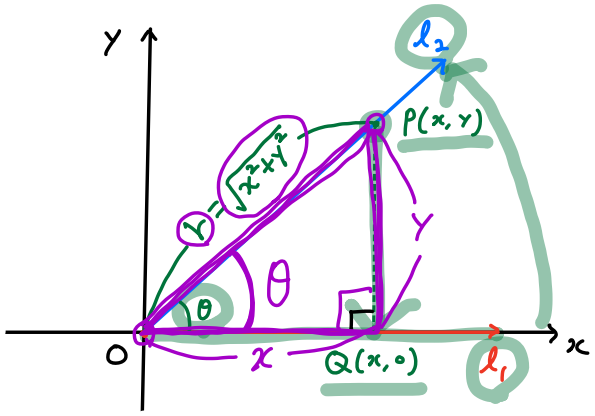
$$= \frac{1}{\sin\theta\cos\theta}$$ ← the same!

$$\csc\theta \cdot \sec\theta = \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$

Hence, $\cot\theta + \tan\theta = \csc\theta \cdot \sec\theta$.

Now, we will define the trigonometric functions to arbitrary angle θ.

When θ is acute ...



Then, $\sin\theta = \dfrac{\text{height}}{\text{hypothenuse}} = \dfrac{y}{r}$, $\csc\theta = \dfrac{\text{hypothenuse}}{\text{height}} = \dfrac{r}{y}$

$\cos\theta = \dfrac{\text{base}}{\text{hypothenuse}} = \dfrac{x}{r}$, $\sec\theta = \dfrac{\text{hypothenuse}}{\text{base}} = \dfrac{r}{x}$

$\tan\theta = \dfrac{\text{height}}{\text{base}} = \dfrac{y}{x}$, $\cot\theta = \dfrac{\text{base}}{\text{height}} = \dfrac{x}{y}$.

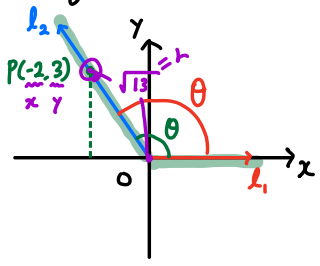
$\Downarrow$

$\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$, $\tan\theta = \dfrac{y}{x}$ (if $x \neq 0$.)

$\csc\theta = \dfrac{r}{y}$ (if $y \neq 0$) , $\sec\theta = \dfrac{r}{x}$ (if $x \neq 0$.), $\cot\theta = \dfrac{x}{y}$. (if $y \neq 0$.)

For arbitrary angle θ, this is the way how we define the trigonometric function !

Ex



$\sin\theta = \dfrac{y}{r} = \dfrac{3}{\sqrt{13}}$ $\csc\theta = \dfrac{r}{y} = \dfrac{\sqrt{13}}{3}$

$\cos\theta = \dfrac{x}{r} = \dfrac{-2}{\sqrt{13}}$ $\sec\theta = \dfrac{r}{x} = \dfrac{\sqrt{13}}{-2} = -\dfrac{\sqrt{13}}{2}$

$\tan\theta = \dfrac{y}{x} = \dfrac{3}{-2} = -\dfrac{3}{2}$ $\cot\theta = \dfrac{x}{y} = \dfrac{-2}{3} = -\dfrac{2}{3}$

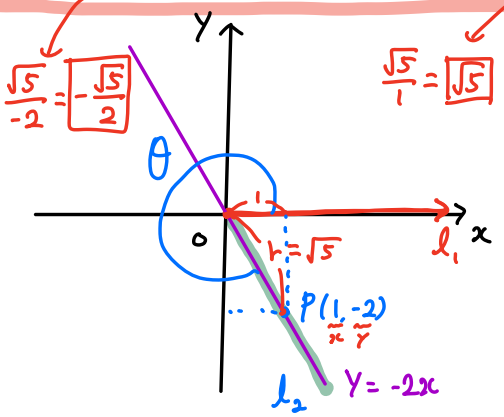
Ex An angle θ is in standard position, and its terminal side lies in quadrant IV on the line $y = -2x$. Find the values of the trigonometric functions of θ.

Recall:

$$\sin\theta = \frac{y}{r} = \frac{-2}{\sqrt{5}}, \quad \cos\theta = \frac{x}{r} = \frac{1}{\sqrt{5}}, \quad \tan\theta = \frac{y}{x} \ (\text{if } x \neq 0) \quad \left(= \frac{-2}{1} = -2\right)$$

$$\csc\theta = \frac{r}{y} \ (\text{if } y \neq 0), \quad \sec\theta = \frac{r}{x} \ (\text{if } x \neq 0), \quad \cot\theta = \frac{x}{y} \ (\text{if } y \neq 0)$$

$\frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$ $\qquad$ $\frac{\sqrt{5}}{1} = \sqrt{5}$



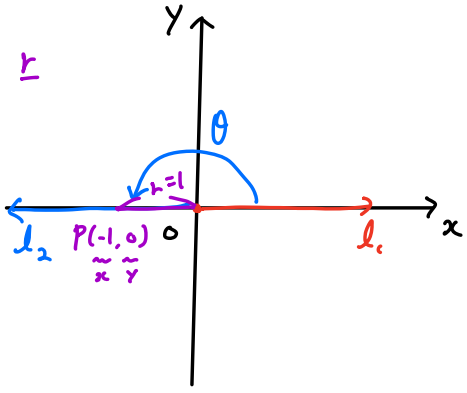
$x = 1, \ y = -2, \ r = \sqrt{5}$

Hence, $\sin\theta = \frac{-2}{\sqrt{5}}$ $\qquad$ $\csc\theta = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$

$\cos\theta = \frac{1}{\sqrt{5}}$ $\qquad$ $\sec\theta = \frac{\sqrt{5}}{1} = \sqrt{5}$

$\tan\theta = \frac{-2}{1} = -2$ $\qquad$ $\cot\theta = \frac{1}{-2} = -\frac{1}{2}$

Ex If $\theta = \pi$, find the values of the trigonometric functions of θ.



$x = -1, \ y = 0, \ r = 1$

$\sin\theta = \frac{y}{r} \Rightarrow \sin\pi = \frac{0}{1} = 0$ $\qquad$ $\csc\theta = \frac{r}{y} \Rightarrow \frac{1}{0}$, so $\csc\pi$ is undefined

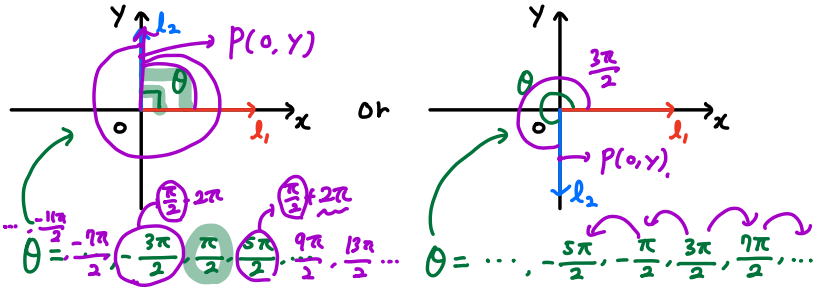
$\cos\theta = \frac{x}{r} \Rightarrow \cos\pi = \frac{-1}{1} = -1$ $\qquad$ $\sec\theta = \frac{r}{x} \Rightarrow \sec\pi = \frac{1}{-1} = -1$

$\tan\theta = \frac{y}{x} \Rightarrow \tan\pi = \frac{0}{-1} = 0$ $\qquad$ $\cot\theta = \frac{x}{y} \Rightarrow \frac{-1}{0}$, so $\cot\pi$ is undefined

* $\sin\theta$ and $\cos\theta$ are defined for every angle.

* $\tan\theta$ ($\frac{y}{x}$) and $\sec\theta$ ($\frac{r}{x}$) are undefined if $x = 0$.

This happens when l_2 is on the y-axis.

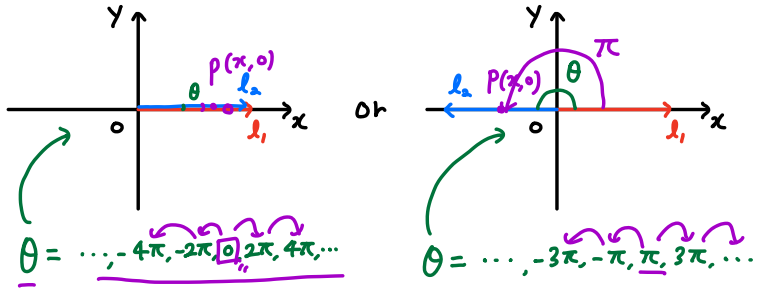


Hence $\tan\theta$ and $\sec\theta$ are defined for every angles except

$$\theta = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$

* $\cot\theta$ ($\frac{x}{y}$) and $\csc\theta$ ($\frac{r}{y}$) are undefined if $y = 0$.

This happens when l_2 is on the x-axis.



Hence $\cot\theta$ and $\csc\theta$ are defined for every angles except

$$\theta = \cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots$$