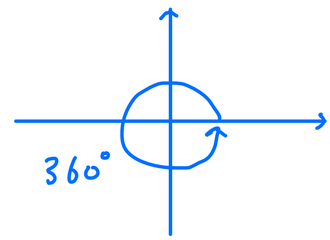
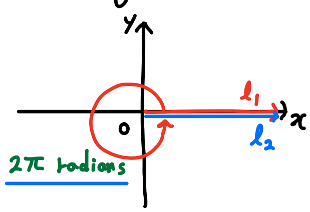
(Section 6.1 Continued)

Unfortunately, in Calculus, we usually use different measure to represent angles.



New measure "radian":

If the ray ℓ_1 turn around once in the counterclockwise direction, the angle determined by the two rays is "2π radians."



Hence, 2π radians $= 360°$ $\Rightarrow$ $\left\{ \begin{array}{l} 1 \text{ radian} \overset{\checkmark}{=} \left(\dfrac{180}{\pi}\right)^\circ \\ 1^\circ \overset{\checkmark}{=} \left(\dfrac{\pi}{180}\right) \text{ radian.} \end{array} \right\}$

Ex (Radian to degree / Degree to radian)

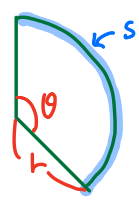
1) $\frac{2}{3}\pi$ radian $= \frac{2}{\cancel{3}_1}\cancel{\pi}^{1} \cdot \frac{\cancel{180}^{60}}{\cancel{\pi}_1} = \boxed{120^\circ}$

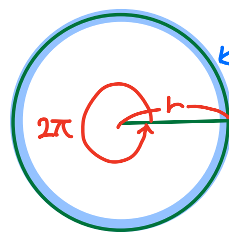
2) $216^\circ = \cancel{216}^{6} \times \frac{\pi}{\cancel{180}_5} = \boxed{\frac{6\pi}{5} \text{ radian}}$

Arc length formula

If an arc of length s on a circle of radius r subtends a central angle of radian measure θ, then

$$\boxed{s = r\theta}$$





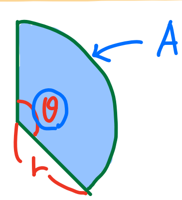
$2\pi r$: circumference of a circle with radius r.

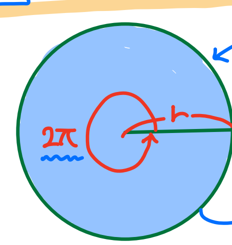
$\boxed{2\pi r} = \underset{\text{radius}}{r} \cdot \underset{\text{central angle}}{2\pi}$

* An arc is proposional to a central angle

Circular sector area formula.

If θ is the radian measure of a central angle of a circle of radius r and if A is the area of the circular sector determined by θ, then $\boxed{A = \frac{1}{2}r^2\theta}$



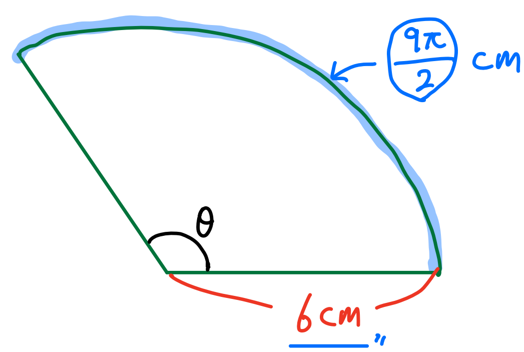


$\boxed{\pi r^2}$: area of a disk with radius r.

$A = \frac{1}{2} \cdot r^2 \cdot 2\pi = \pi r^2$

* An area of the circula sector is proposional to a central angle.

Ex Find the angle θ and the area of the following circular sector:



$$\boxed{S = r\theta}$$

$$\frac{9\pi}{2} = 6 \cdot \theta$$

$$\Rightarrow \theta = \frac{9\pi}{12} = \frac{3\pi}{4} \text{ radian } (= 135^\circ)$$

$$A = \frac{1}{2} \cdot r^2 \cdot \theta = \frac{1}{2} (6\,\text{cm})^2 \cdot \frac{3\pi}{4}$$

$$= \frac{1}{2} \cdot \cancel{36}^{\,9} \cdot \frac{3\pi}{\cancel{4}_{\,1}}$$

$$= \boxed{\frac{27\pi}{2}\ \text{cm}^2}$$

Section 6.2. Trigonometric Functions of Angles.

Six Trigonometric functions : Sine (sin) cosecant (csc)

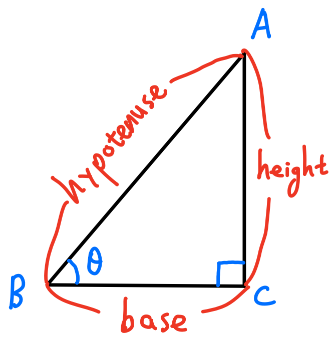
cosine (cos) secant (sec)

tangent (tan) cotangent. (cot)

We first define these function when input (θ) is an acute angle. : $0^\circ < \theta < 90^\circ$

0 radian $< \theta < \frac{\pi}{2}$ radian

For any acute angle θ, consider any right triangle $\triangle ABC$ such that $\angle B = \theta$ and $\angle C = 90^\circ$



Then, $\sin\theta = \dfrac{\text{height}}{\text{hypothenuse}}$, $\csc\theta = \dfrac{\text{hypothenuse}}{\text{height}}$

$\cos\theta = \dfrac{\text{base}}{\text{hypothenuse}}$, $\sec\theta = \dfrac{\text{hypothenuse}}{\text{base}}$

$\tan\theta = \dfrac{\text{height}}{\text{base}}$, $\cot\theta = \dfrac{\text{base}}{\text{height}}$

* When θ is an acute angle, by the definition the values of the six trigonometric functions are all positive!

From the definition, we can easily see that $\sin\theta$ and $\csc\theta$, $\cos\theta$ and $\sec\theta$, $\tan\theta$ and $\cot\theta$ are reciprocal of each other, respectively.

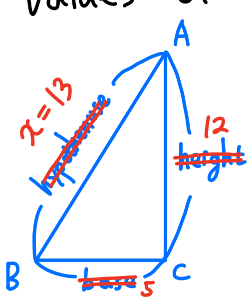
Hence, we have the following :

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta}, \quad \cos\theta = \frac{1}{\sec\theta}, \quad \tan\theta = \frac{1}{\cot\theta},$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

<u>Ex</u> If θ is an acute angle and $\tan\theta = \frac{12}{5}$. find the values of the trigonometric functions of θ.



By definition, $\tan\theta = \frac{\text{height}}{\text{base}} = \frac{12}{5} \longrightarrow$ choose (height) = 12, (base) = 5.

Let x = (hypothenuse).

By the Pythagorean theorem, $x^2 = 5^2 + 12^2 = 25 + 144 = 169 = 13^2$. $x^2 = 13^2 \Rightarrow x = 13$.

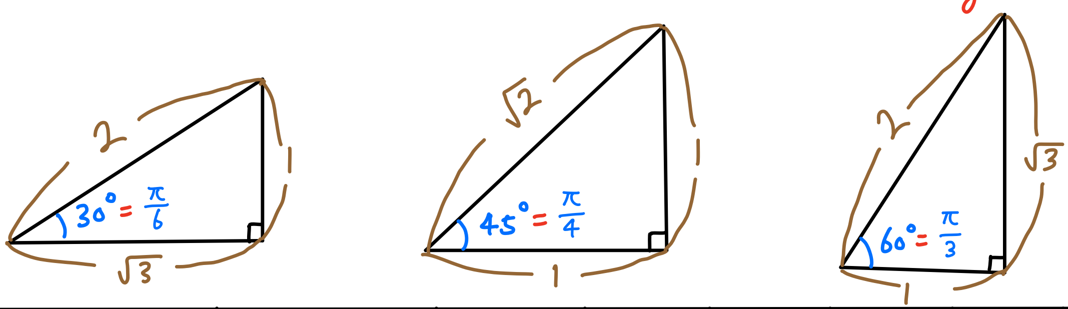
$$\boxed{\begin{array}{l} \sin\theta = \frac{12}{13}, \quad \cos\theta = \frac{5}{13}, \quad \tan\theta = \frac{12}{5} \\ \csc\theta = \frac{13}{12}, \quad \sec\theta = \frac{13}{5}, \quad \sec\theta = \frac{5}{12} \end{array}}$$

* Three important triangles.

If you remember these triangles, you can easily remember the values of the trigonometric functions when $\theta = \frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$

30° 45° 60°

The most important acute angles!



θ (radian)	θ (degree)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$

The following is the most fundamental and important identities! (You should get used to them!)

Fundamental Identities

(1) The reciprocal identities : $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$

(2) The tangent and cotangent identities : $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(3) The Pythagorean identities : ① $\sin^2\theta + \cos^2\theta = 1$

② $1 + \tan^2\theta = \sec^2\theta$

③ $1 + \cot^2\theta = \csc^2\theta$

Proof of (2)

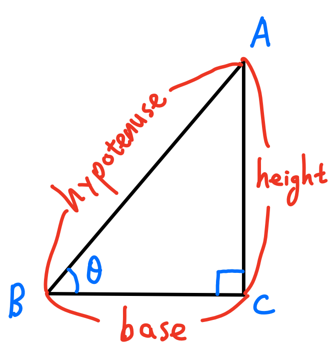
$* \ \dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{ad}{bc}$

: Recall that $\sin\theta = \dfrac{\text{height}}{\text{hypothenuse}}$, $\cos\theta = \dfrac{\text{base}}{\text{hypothenuse}}$, $\tan\theta = \dfrac{\text{height}}{\text{base}}$.

$\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{\text{height}}{\text{hypothenuse}}}{\frac{\text{base}}{\text{hypothenuse}}} = \dfrac{(\text{height})\cdot\cancel{(\text{hypothenuse})}}{\cancel{(\text{hypothenuse})}\cdot(\text{base})} = \dfrac{\text{height}}{\text{base}} = \tan\theta$. Hence, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

Also, $\cot\theta = \dfrac{1}{\tan\theta} = (\tan\theta)^{-1} = \left(\dfrac{\sin\theta}{\cos\theta}\right)^{-1} = \dfrac{\cos\theta}{\sin\theta}$. Hence, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Proof of (3) : Recall the Pythagorean theorem



$$(\text{base})^2 + (\text{height})^2 = (\text{hypotenuse})^2 \quad \cdots (*)$$

Divide $(*)$ by $(\text{hypothenuse})^2$ $\Rightarrow \dfrac{(\text{base})^2}{(\text{hypothenuse})^2} + \dfrac{(\text{height})^2}{(\text{hypothenuse})^2} = 1$

$\Rightarrow \left(\dfrac{\text{base}}{\text{hypothenuse}}\right)^2 + \left(\dfrac{\text{height}}{\text{hypothenuse}}\right)^2 = 1$

$\Rightarrow \boxed{\cos^2\theta + \sin^2\theta = 1}$

Divide $(*)$ by $(\text{base})^2$

$\Rightarrow 1 + \dfrac{(\text{height})^2}{(\text{base})^2} = \dfrac{(\text{hypothenuse})^2}{(\text{base})^2}$

$\Rightarrow 1 + \left(\dfrac{\text{height}}{\text{base}}\right)^2 = \left(\dfrac{\text{hypothenuse}}{\text{base}}\right)^2$

$\Rightarrow \boxed{1 + \tan^2\theta = \sec^2\theta}$

Divide $(*)$ by $(\text{height})^2$

$\Rightarrow \dfrac{(\text{base})^2}{(\text{height})^2} + 1 = \dfrac{(\text{hypothenuse})^2}{(\text{height})^2}$

$\Rightarrow \left(\dfrac{\text{base}}{\text{height}}\right)^2 + 1 = \left(\dfrac{\text{hypothenuse}}{\text{height}}\right)^2$

$\Rightarrow \boxed{\cot^2\theta + 1 = \csc^2\theta}$

$*$ For any integer n and $n \neq -1$,
$(\sin\theta)^n$ is denoted by $\sin^n\theta$.
$(\sin\theta)^{-1}$ is denoted by $(\sin\theta)^{-1}$,
not by $\sin^{-1}\theta$.