Section 5.4 Continued.

Solving an equation involving exponential / logarithm in one variable

Ex On the Richter scale, the magnitude R of an earthquake of intensity I is given by $R = \log \frac{I}{I_0}$ where I_0 is a certain minimum intensity.

1) If the intensity (I) of an earthquake is $10000 I_0$, find R.

2) Express I in terms of R and I_0

1) $R = \log \frac{I}{I_0}$ ← $10{,}000 I_0$

$I = 10{,}000 I_0$: Given   Recall $\log = \log_{10}$ $\log_a a^x = x$

$$\Rightarrow R = \log \frac{10{,}000 \cancel{I_0}}{\cancel{I_0}} = \log 10{,}000 = \log_{10} 10{,}000 = \log_{10} 10^4 = \boxed{4}$$

2) $R = \log \frac{I}{I_0}$

$R = \log_{10}\left(\frac{I}{I_0}\right)$ Recall: $y = \log_a x$ if $x = a^y$

$$\left(\frac{I}{I_0}\right) = 10^R \xrightarrow[I_0]{\text{multiply}} \boxed{I = I_0 \cdot 10^R}$$

Ex Suppose $\$P_0$ is deposited in a money market account that pays interest at a rate of 10% per year compounded continuously. How long will it take to have $\$5 \cdot P_0$ in the account?

principal

$r = 0.10$

t years.

$\$A$

"Recall"

- Continuously Compounded Interest Formula

$$A = Pe^{rt},$$ where P = principal

r = annual interest rate expressed as a decimal

t = number of years P is invested

A = amount after t years.

$$A = Pe^{rt}$$

$$\frac{5 \cdot P_0}{P_0} = \frac{P_0 \cdot e^{0.1 \cdot t}}{P_0}$$

$y = a^x$ if $x = \log_a y$

$$5 = e^{0.1 \cdot t} \longrightarrow \text{"}\boxed{0.1 \cdot t = \log_e 5}\text{"}$$

$\times 10$

$$10 \cdot 0.1 \cdot t = 10 \cdot \log_e 5$$

$\log_e = \ln$

$$t = 10 \cdot \log_e 5$$

$$= \boxed{10 \cdot \ln 5} \text{ years.}$$

Section 5.5. Properties of Logarithms.

Law of Logarithms

If u and w denote positive real numbers, then

(1) $\log_a(uw) = \log_a u + \log_a w$

(2) $\log_a\left(\frac{u}{w}\right) = \log_a u - \log_a w$

(3) $\log_a(u^c) = c\log_a u$ for every real number c.

Proof: (1) $a^{\log_a(uw)} = uw$ and $a^{\log_a u + \log_a w} = a^{\log_a u} \cdot a^{\log_a w} = u \cdot w$.

Hence, $a^{\log_a(uw)} = a^{\log_a u + \log_a w}$. $\Rightarrow \log_a(uw) = \log_a u + \log_a w$.

(2) $a^{\log_a\left(\frac{u}{w}\right)} = \frac{u}{w}$ and $a^{\log_a u - \log_a w} = \frac{a^{\log_a u}}{a^{\log_a w}} = \frac{u}{w}$.

Hence, $a^{\log_a\left(\frac{u}{w}\right)} = a^{\log_a u - \log_a w} \Rightarrow \log_a\left(\frac{u}{w}\right) = \log_a u - \log_a w$.

(3) $a^{\log_a(u^c)} = u^c$ and $a^{c\log_a u} = \left(a^{\log_a u}\right)^c = u^c$.

Hence, $a^{\log_a(u^c)} = a^{c\log_a u} \Rightarrow \log_a(u^c) = c\log_a u$

$\log_{10} \to \log \qquad \log_e \to \ln$

When $a = 10$ or e, the above laws become the following:

(1) $\log(uw) = \log u + \log w$

(2) $\log\left(\frac{u}{w}\right) = \log u - \log w$

(3) $\log(u^c) = c \log u$

for every real number c.

(1) $\ln(uw) = \ln u + \ln w$

(2) $\ln\left(\frac{u}{w}\right) = \ln u - \ln w$

(3) $\ln(u^c) = c \cdot \ln u$

for every real number c.

Using these properties, we will solve several problems.

Ex1 Express $\ln \frac{x^4 \cdot \sqrt[3]{y}}{z}$ in terms of logarithms of x, y, and z.

$\ln x$, $\ln y$, and $\ln z$

$$\ln\left(\frac{x^4 \cdot \sqrt[3]{y}}{z}\right) = \ln(x^4 \cdot \sqrt[3]{y}) - \ln z$$

$$= \ln x^4 + \ln \sqrt[3]{y} - \ln z$$

$$= 4 \ln x + \ln y^{\frac{1}{3}} - \ln z$$

$$= \boxed{4 \ln x + \frac{1}{3} \ln y - \ln z}$$

Ex 2 Express as one logarithm: $\frac{1}{5}\log_a(x^2+x) - 2\log_a y - 3\log_a z$

$$= \log_a(x^2+x)^{\frac{1}{5}} - \log_a y^2 - \log_a z^3$$

$$= \log_a\left(\frac{(x^2+x)^{\frac{1}{5}}}{y^2}\right) - \log_a z^3$$

$$\left(\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}\right)$$

$$= \log_a\left(\frac{\frac{(x^2+x)^{\frac{1}{5}}}{y^2}}{\frac{z^3}{1}}\right)$$

$$= \boxed{\log_a\left(\frac{(x^2+x)^{\frac{1}{5}}}{y^2z^3}\right)} = \boxed{\log_a\left(\frac{\sqrt[5]{x^2+x}}{y^2z^3}\right)}$$

Ex 3 Solve $\log_7(3x+8) = \log_7 2 + \log_7 13$.

6

$3\cdot 6+8$
$=18+8=\boxed{26}$

$\log_7(2\cdot 13)$

$$\log_7(3x+8) = \log_7 26$$

$$\Rightarrow 3x+8=26,\quad 3x=18,\quad \boxed{x=6}$$

Ex 4 Solve $\log_3 x + \log_3(x+6) = 3$

$-9 \qquad -9$

$\boxed{y=\log_a x \text{ if } x=a^y}$

$\cancel{\log_3(-9)}$

$$\log_3 x(x+6) = 3 \longleftrightarrow x(x+6) = 3^3$$

$\boxed{\text{-9 is not in the domain of } f(x)=\log_3 x.}$

$$x^2+6x=27$$
$$x^2+6x-27=0.$$
$$(x+9)(x-3)=0.$$

Z.F.T.

$$x+9=0 \text{ or } x-3=0.$$
$$\cancel{x=-9} \text{ or } \boxed{x=3}.$$

Ex 5 Solve $\ln(x-3) - \ln(x+2) = \ln 5 - \ln 3$

$$\ln \frac{x-3}{x+2} = \ln \frac{5}{3}$$

$$\Rightarrow \frac{x-3}{x+2} = \frac{5}{3},$$

$$\Rightarrow 3(x-3) = 5(x+2)$$

$$\Rightarrow 3x - 9 = 5x + 10 \rightarrow 2x = -19, \ \cancel{x = -\frac{19}{2}}$$

$$\ln\left(-\frac{19}{2} - 3\right) = \ln\left(-\frac{25}{2}\right)$$

it is not in the domain!

$\Rightarrow$ $\boxed{\text{No solution!}}$

Ex 6 Sketch the graph of $y = \log_2(8x)$

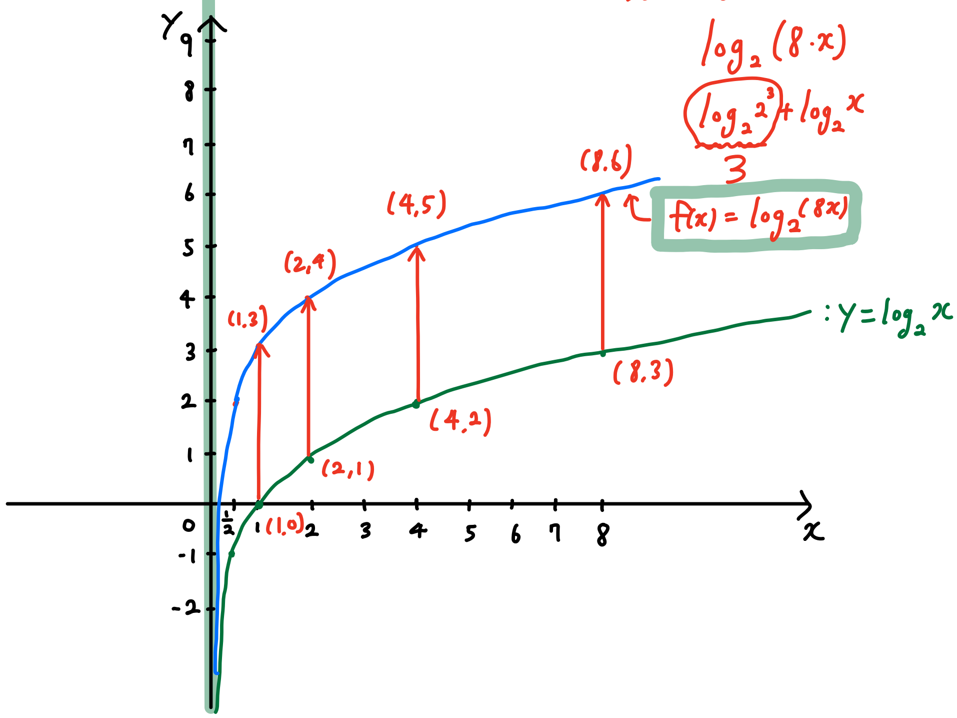
$y = 3 + \log_2 x$

$\log_2(8 \cdot x)$

$\log_2 2^3 + \log_2 x$

3

$\boxed{f(x) = \log_2(8x)}$

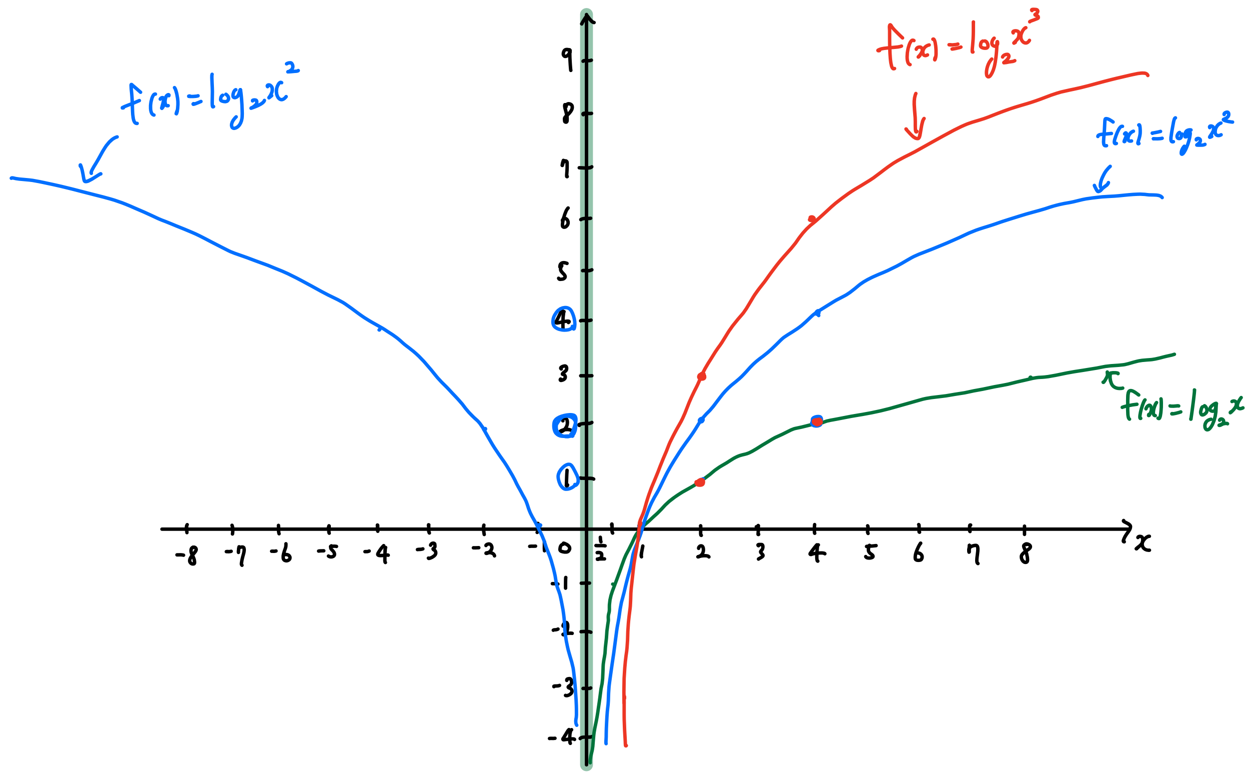


$f(-x) = \log_2 (-x)^2 = \log_2 x^2 = f(x)$: even function!

Domain : $(-\infty, 0) \cup (0, \infty)$ Domain : $(0, \infty)$

Ex 7 Sketch the graph of $f(x) = \log_2 x^2$ and $f(x) = \log_2 x^3$

$= 2 \cdot \log_2 x$ if $x > 0$. $= 3 \cdot \log_2 x$



Ex 8 Solve $k \log_a m = \log_a n - \log_a P$ for P

$\log_a (m^k) = \log_a \left(\frac{n}{P}\right) \Rightarrow m^k = \frac{n}{P} \overset{\times P}{\Rightarrow} P \cdot m^k = n$

$\Rightarrow \boxed{P = \frac{n}{m^k}}$