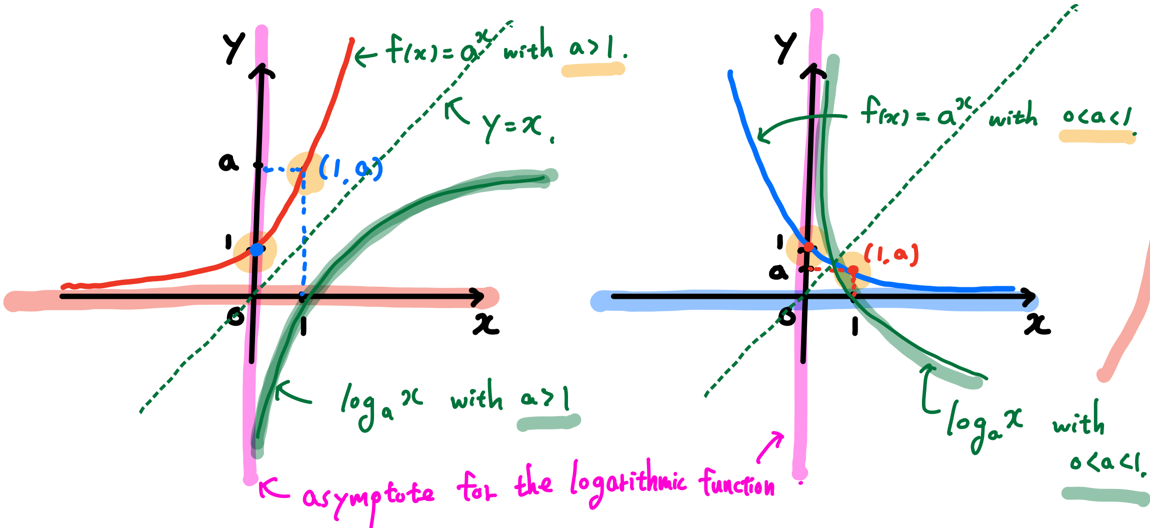
Section 5.4. Logarithmic Functions.

Recall: Exponential Functions :

Domain	: $(-\infty, \infty) = \mathbb{R}$
Range	: $(0, \infty)$

In Section 5.2, we learned that $f(x) = a^x$ is one-to-one.



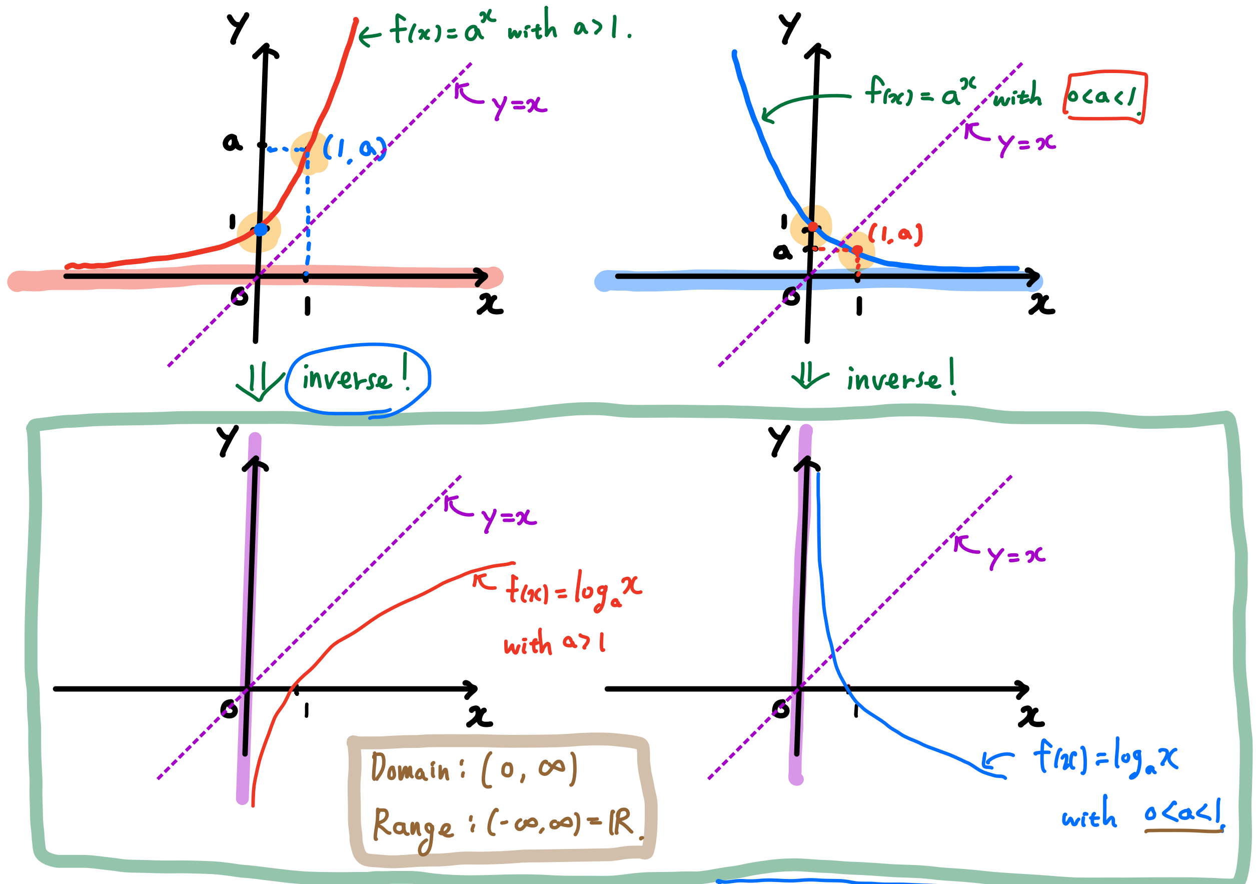
Since the exponential function is one-to-one, we can think of its inverse function.

→ "logarithmic function."

it is an inverse function of $f(x) = a^x$.

For any $a > 0$ and $a \neq 1$, $f(x) = \log_a x$ is called "Logarithmic function with base a".

Graphs of the logarithmic functions.



Recall If $f(x)$ is one-to-one, then $\begin{cases} f(f^{-1}(x)) = x \\ f^{-1}(f(x)) = x \end{cases}$

Since $f(x) = a^x$ and $f(x) = \log_a x$ are inverse to each other,

we have the following alternative definition of $\log_a$:

For any $a > 0$ and $a \neq 1$, $n = \log_a m$ if $m = a^n$

for every $m > 0$ and every real number n.

logarithmic form

exponential form

$n = \log_a m$ if $m = a^n$: Definition.

Basic properties of the logarithm.

(1) $\log_a 1 = 0$ $\qquad 1 = a^0 \Rightarrow 0 = \log_a 1$ $\qquad$ Ex $\log_{11} 1 = 0$

(2) $\log_a a = 1$ $\qquad a = a^1 \Rightarrow 1 = \log_a a$ $\qquad$ Ex $\log_5 5 = 1$

(3) $\log_a a^x = x$ $\qquad a^x = a^x \Rightarrow x = \log_a a^x$ $\qquad$ Ex $\log_4 2 = \log_4 4^{\frac{1}{2}} = \frac{1}{2}$

(4) $a^{\log_a x} = x$ $\qquad$ Ex $3^{\log_3 7} = 7$.

Recall that $f(f^{-1}(x)) = x$

Let $f(x) = a^x \to f^{-1}(x) = \log_a x$

$$x = f(f^{-1}(x)) = f(\log_a x) = a^{\log_a x}$$

Recall that the graph of the logarithmic function is either increasing (if $a > 1$) or decreasing (if $0 < a < 1$).

$\Rightarrow$ Hence, the logarithmic function $f(x) = \log_a x$ is one-to-one. !

Thus, for real numbers x_1 and x_2, the following equivalent conditions are satisfied:

(1) If $x_1 \neq x_2$, then $\log_a x_1 \neq \log_a x_2$. $\quad$ ($f(x_1) \neq f(x_2)$)

(2) If $\log_a x_1 = \log_a x_2$, then $x_1 = x_2$. $\quad$ ($f(x_1) = f(x_2)$)

Using the above property and the definition of the logarithmic function, we can solve various equations involving logarithms.

* Domain of the logarithmic function is $(0, \infty)$

Ex ① Solve $\log_3(2x+7) = \log_3(5x-2)$

$\Rightarrow 2x+7 = 5x-2$

$\Rightarrow 3x = 9$, $\boxed{x=3}$

$\log_3(2\cdot3+7) = \log_3(5\cdot3-2)$

$\log_3 13 = \log_3 13 \Rightarrow \boxed{x=3}$

② Solve $\log_3(2x-7) = \log_3(5x+2)$

$\Rightarrow 2x-7 = 5x+2$

$\Rightarrow 3x = -9$, $\boxed{x=-3}$

$\log_3(2\cdot(-3)-7) = \log_3(5\cdot(-3)+2)$

$\log_3(-13) = \log_3(-13)$: -13 is not in the domain!

$\Rightarrow$ No solution.

③ Solve $\log_3(x-2) = 4$ (83)

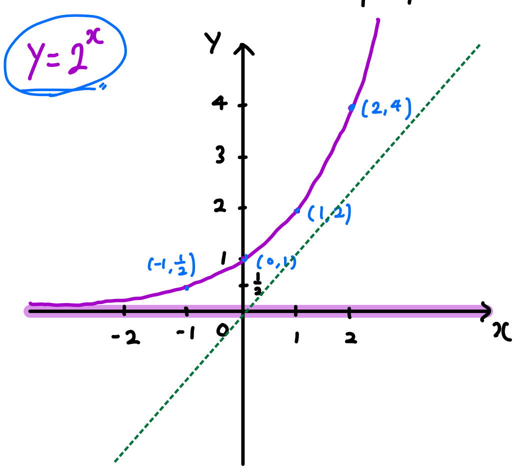
$m = \log_a n$ if $n = a^m$

$4 = \log_3(x-2)$ if $(x-2) = 3^4 \rightarrow x-2 = 81$

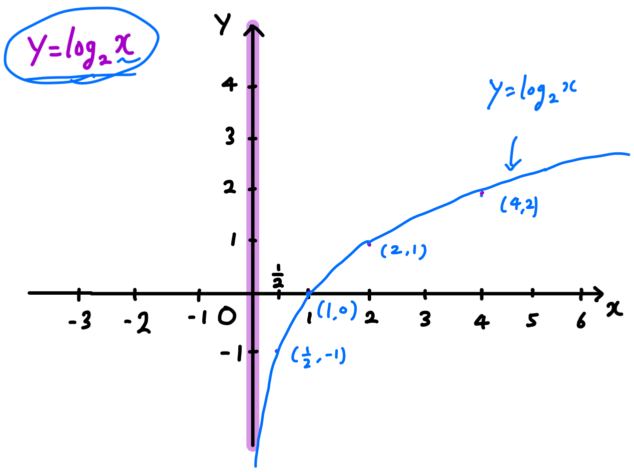
$\rightarrow \boxed{x = 83}$

Now, let us play with the graph of logarithmic function.

$Y = 2^x$



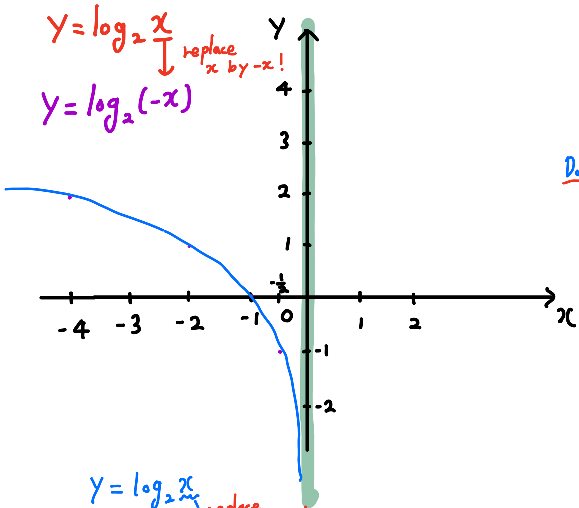
$Y = \log_2 x$

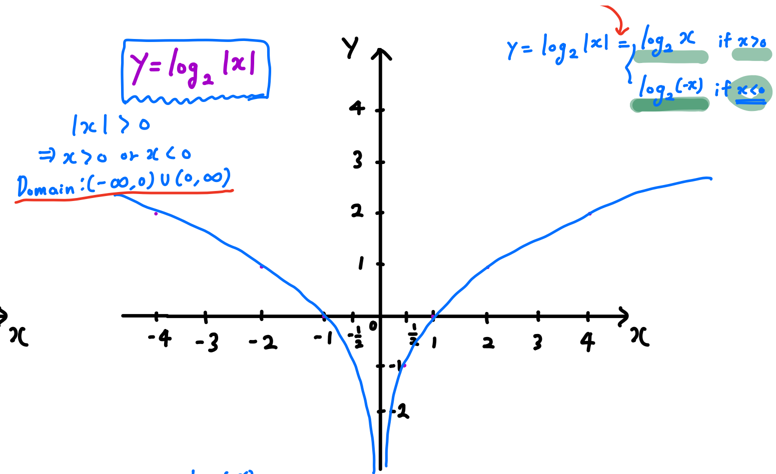


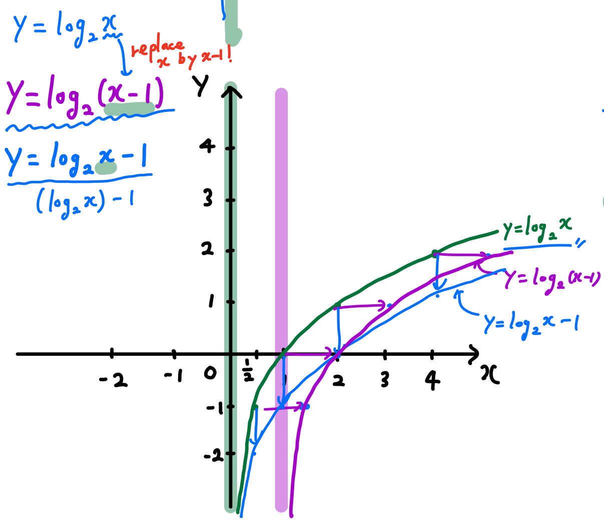
$x \rightarrow -x$,

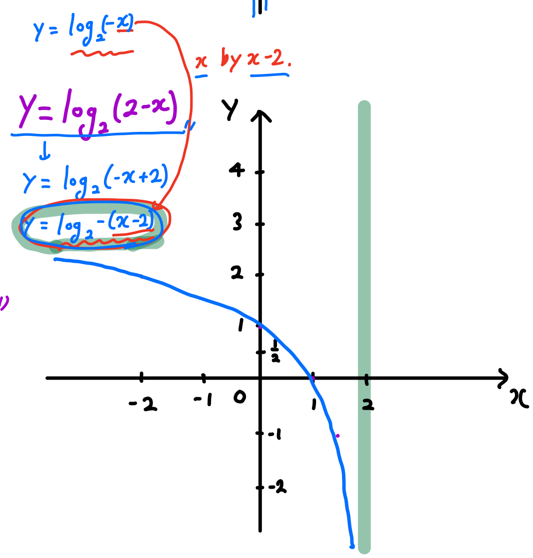
$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

piecewise-defined function!









When the base $a = 10$, $\log_{10} x$ is called "common logarithm".

$\log x$ $\quad\quad$ $\log = \log_{10}$

When the base $a = e$, $\log_e x$ is called "natural logarithm".

$\ln x$ $\quad\quad$ $\ln = \log_e$

	$a = 10$	$a = e$
$\log_a 1 = 0$	$\log 1 = 0$	$\log_e 1 = \ln 1 = 0$
$\log_a a = 1$	$\log 10 = 1$	$\log_e e = \ln e = 1$
$\log_a a^x = x$	$\log 10^x = x$	$\ln e^x = x$
$a^{\log_a x} = x$	$10^{\log x} = x$	$e^{\ln x} = x$

$e^{\ln 2} = 2$

useful when we convert to base e expression!

Ex (1) $y = 2^x = (e^{\ln 2})^x = e^{\ln 2 \cdot x}$

(2) $y = x^2 = (e^{\ln x})^2 = e^{2 \cdot \ln x}$.