(Section 5.2 Continued)

In many cases, the interest is compounded more than once a year!

Ex Suppose $P = \$1,000$, the interest rate (compound interest) is 20% and the interest is compounded every 3 months.

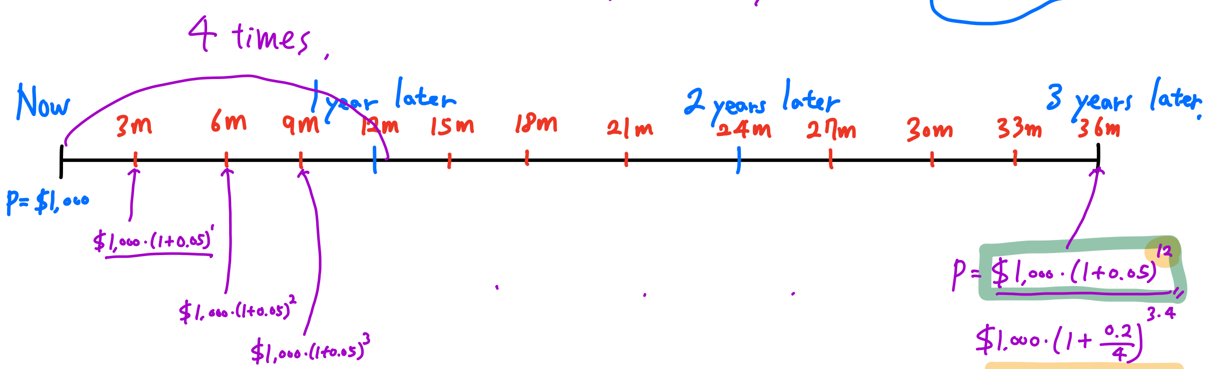
annual interest.

4 times a year.

20% after 12 months

$\downarrow \frac{1}{4}$ $\quad \downarrow \times\frac{1}{4}$

5% after 3 months.



- Compound Interest Formula

Suppose P is the principal (= amount of money that was initially invested)

and r is the annual interest rate.

Also, suppose the interest is compounded n times a year.

Then after t years, the total amount of money is $P \cdot \left(1+\frac{r}{n}\right)^{nt}$

$1,000 \cdot (1.05)^{12} = 1,000 \cdot \left(1+\frac{0.20}{4}\right)^{4\cdot 3}$

Application : Finding an exponential model.

Ex In 1951, the population of the city A was 23,500.

In 1976, the popluation of the city A became 470,000.

Find a simple exponential function of the form $\boxed{y = ab^t}$

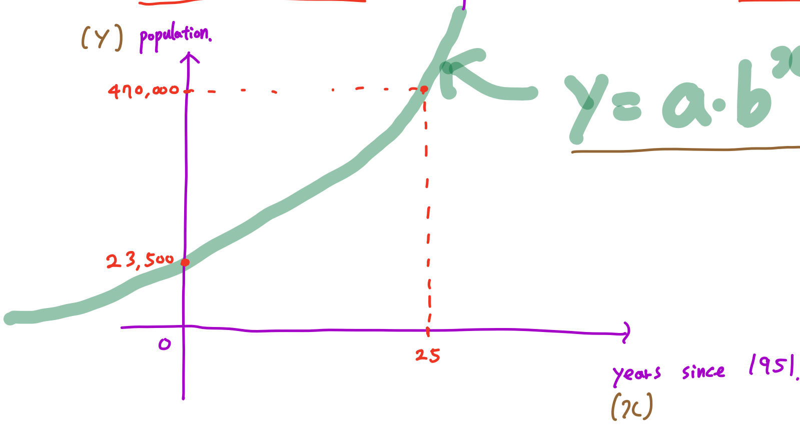
that models the population for 1951 – 1976.

(use the fact that $\sqrt[25]{20} \approx 1.1273$)

In 1951, Population was 23,500

0 year sin 1951

After 25 years, Population became 470,000.



$y = a \cdot b^x$: $(0, 23,500)$ and $\boxed{(25, 470,000)}$ are on the graph!

$\Rightarrow$ Replace $x = 0$, $y = 23,500$

: $23,500 = a \cdot b^0$, $\boxed{a = 23,500}$

$\Rightarrow$ $y = 23,500 \cdot b^x$

$\Rightarrow$ Replace $x = 25$, $y = 470,000$

: $\dfrac{470,000}{23,500} = \dfrac{23,500 \cdot b^{25}}{23,500}$

$\Rightarrow$ $20 = b^{25}$, $b = \sqrt[25]{20} \approx 1.1273$

$\Rightarrow$ $\boxed{y = 23,500 \cdot (1.1273)^t}$

Section 5.3. The Natural Exponential Function.

Let us observe the quantity $\left(1+\frac{1}{n}\right)^n$ for positive integer n.

n	1	2	3	4	$\cdots$	100	$\cdots$
$\left(1+\frac{1}{n}\right)^n$	$\left(1+\frac{1}{1}\right)^1$	$\left(1+\frac{1}{2}\right)^2$	$\left(1+\frac{1}{3}\right)^3$	$\left(1+\frac{1}{4}\right)^4$	$\cdots$	$\left(1+\frac{1}{100}\right)^{100}$	$\cdots$
	$(1+1)^1 = 2$	$\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2.25$	$\cdots$				

It seems the quantity $\left(1+\frac{1}{n}\right)^n$ is getting closer and closer to a certain real number!

We call this number a "natural constant", and denote it by "e".

Later (not in this course), you can prove that this is true.

$$\left(1+\frac{1}{n}\right)^n \longrightarrow \underbrace{2.71828\cdots}_{=e} \text{ as } n \to \infty$$

irrational number!

Exponential function is $f(x) = a^x$ for some $a > 0,\ a \neq 1$.

Natural expential function

is an exponential function whose base is e! : $\boxed{f(x) = e^x}$

Why "natural"?

You will learn that the natural exponential function is the unique function whose derivative is the same as the function itself!

"$f(x) = e^x$" $\longrightarrow$ "$f'(x) = e^x$"

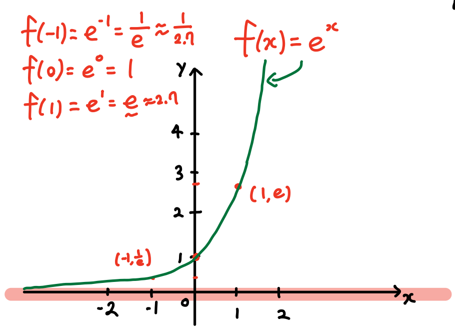
(You will see this in M211)

We already know how to plot the natural exponential function and how to solve equations involving the natural exponential function.

$f(-1) = e^{-1} = \frac{1}{e} \approx \frac{1}{2.7}$

$f(0) = e^0 = 1$

$f(1) = e^1 = e \approx 2.7$



Ex Find zeros of $f(x) = x^3(-2e^{-2x}) + 3x^2e^{-2x}$

Set $f(x) = 0$

$x^3 \cdot (-2e^{-2x}) + 3x^2 \cdot e^{-2x} = 0$

$-2x^3 \cdot e^{-2x} + 3x^2 \cdot e^{-2x} = 0.$

$(-2x) \cdot x^2 e^{-2x} + 3 \cdot x^2 \cdot e^{-2x} = 0$

$x^2 e^{-2x} \cdot (-2x+3) = 0.$

$\Big\downarrow$ Z.F.T.

$x^2 = 0$ or $\cancel{e^{-2x} = 0}$ or $(-2x+3) = 0.$ $\therefore$ $\boxed{x = 0 \text{ or } x = \frac{3}{2}}$

$x = 0$ $\qquad$ $x = \frac{3}{2}$

$\left(1+\frac{1}{n}\right)^n$ is obtained from the "Compound Interest Formula"

by setting $P=1$, $r=1$, and $t=1$.

$\hookrightarrow P\cdot\left(1+\frac{r}{n}\right)^{nt} \to$ "$\left(1+\frac{1}{n}\right)^n$"

In fact, $e = 2.71828\cdots$ is related to the compound interest via the following theorem

$$P\cdot\left(1+\frac{r}{n}\right)^{nt} \xrightarrow{P=r=t=1} \left(1+\frac{1}{n}\right)^n \longrightarrow e \quad \text{as} \quad n \longrightarrow \infty$$

But n is the number of interest period per year.

Thus "$n \longrightarrow \infty$" can be interpreted as "interest is compounded every seconds"

or "interest is compounded continuously"

- Continuously Compounded Interest Formula

$A = Pe^{rt}$, where P = principal

r = annual interest rate expressed as a decimal

t = number of years P is invested

A = amount after t years.

Ex Suppose \$10,000 is deposited in a money market account that pays interest at a rate of 7% per year compounded continuously. Determine the balance in the account after 8 years.

P. $r = 0.07$ (A) $t = 8$.

$$A = P \cdot e^{rt} = 10{,}000 \cdot e^{0.07 \cdot 8} = \boxed{10{,}000 \cdot e^{0.56}}$$

The following is the more general formula that can be used in many different problems.

Law of Growth (or Decay) Formula

Let q_0 be the value of a quantity q at time $t = 0$.
If q changes instantaneously at a rate proportional to its current value, then $q = q(t) = q_0 \cdot e^{rt}$ where r is the rate.

Ex The population of a city in 1960 was 147,350.

Assuming that the population decrease continuously at a rate of 6% per year, predict the population of the city in the year 2015.

$r = -0.06$.

$t = 55$ year

$$1960 \xrightarrow{\quad} 2015$$

population: 147,350 $= q_0$

$$q = q_0 \cdot e^{-0.06 \cdot 55}$$

$$= \boxed{147{,}350 \cdot e^{-3.3}}$$