Do the Practice Exam 2!

Chapter 5. Inverse, Exponential, and Logarithmic function.

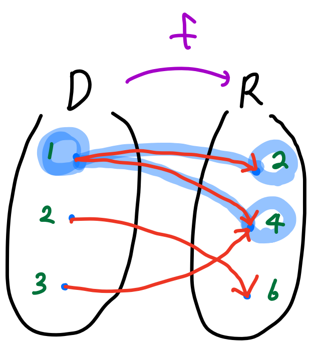
Section. 5.1. Inverse Functions.

One-to-one function

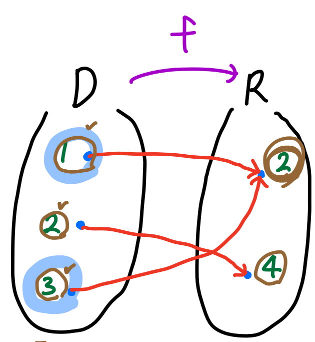
: A function f with domain D and range R is $\boxed{\text{one-to-one function}}$ if either of the following equivalent conditions is satisfied.

(1) Whenever $\overset{1 \neq 3}{a \neq b}$ in D, then $\overset{f(1) \neq f(3)}{f(a) \neq f(b)}$ in R.

(2) Whenever $f(a) = f(b)$ in R, then $a = b$ in D.

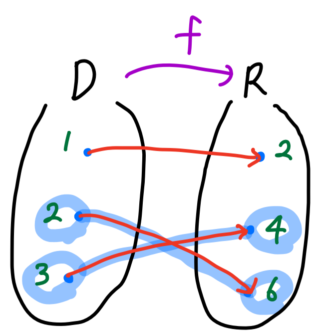


Not a function.



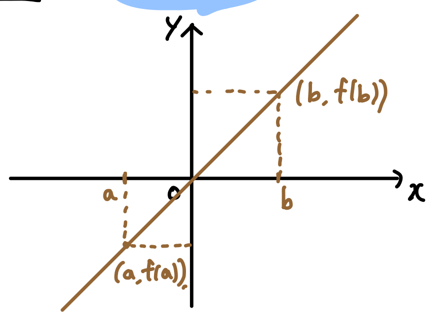
It is a function, which is not one-to-one.

1 and 3 are different, but f(1) and f(3) are the same.



It is a one-to-one function

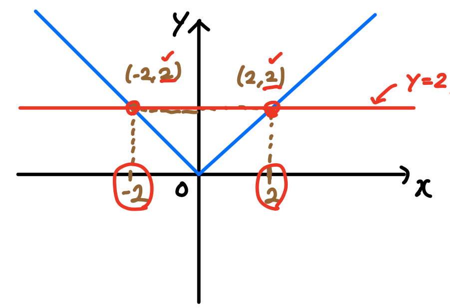
Ex ① $f(x)=x$: one-to-one function



If $a \neq b$, is it possible that $\underbrace{f(a)}_{=a} = \underbrace{f(b)}_{=b}$? No!

Hence $a \neq b \Rightarrow f(a) \neq f(b)$

② $f(x)=|x|$: It is a function that is not one-to-one.



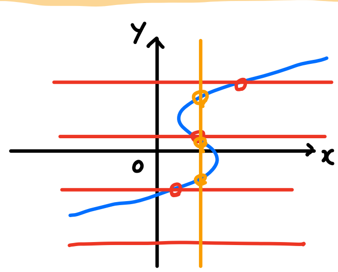
Choose $x=2$ and $x=-2$.

$f(2)=|2|=2$, $\boxed{f(-2)=|-2|=2}$

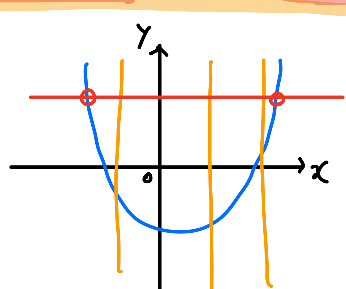
By observing the graph, we can check whether the function is one-to-one or not.

Horizontal line test

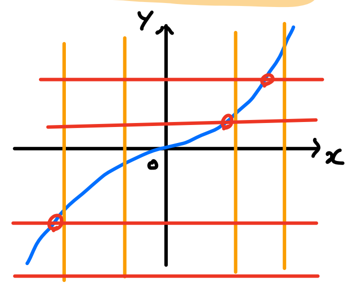
: A function is one-to-one if and only if every horizontal line intersects the graph of f in at most one point.



It is not a graph of a function.

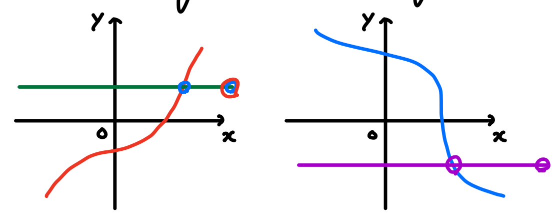


It is a graph of function which is not one-to-one.



It is a graph of one-to-one function.

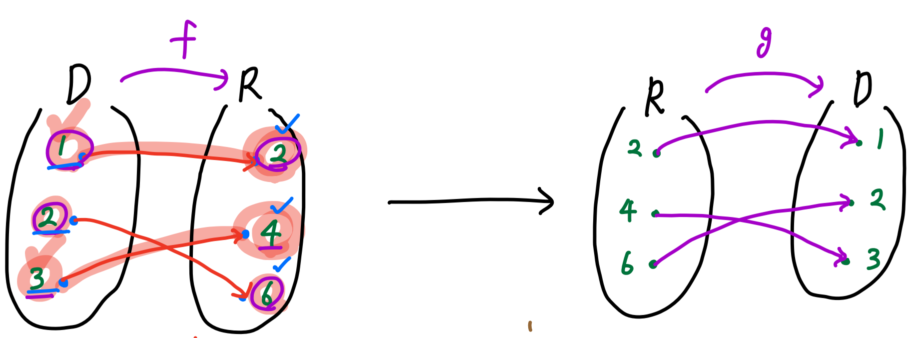
What happens if the function is increasing or decreasing?



From the above observation, we have the following theorem :

> If a function is increasing throughout its domain
> or it is decreasing throughout its domain,
> then the function is one-to-one

Whenever we have a one-to-one function, we can think of another function (called inverse function) as follows.



: the new function.

$f(1) = 2 \longleftrightarrow g(2) = 1$

$f(2) = 6 \longleftrightarrow g(4) = 3$

$f(3) = 4 \longleftrightarrow g(6) = 2$

$f(g(2)) = f(1) = 2$

$g(f(1)) = g(2) = 1$

Formal definition of Inverse Function

Let f be a one-to-one function with domain D and range R.
A function g with domain R and range D is the "inverse function of f", provided the following condition is true for every x in D and every y in R: $\boxed{y = f(x) \text{ if and only if } x = g(y)}$

Theorem on Inverse Functions.

Let f be a one-to-one function with domain D and range R. If g is a function with domain R and range D, then g is inverse function of f if and only if both of the following conditions are true: (1) $\boxed{g(f(x)) = x}$ for every x in D.

(2) $\boxed{f(g(y)) = y}$ for every y in R.

Notation

: the inverse function of a one-to-one function f is usually denoted by f^{-1}.

* $3^{-1} = \frac{1}{3}$. but $f^{-1}(y) \neq \frac{1}{f(y)}$ $[f(y)]$

* domain of f.

x	1	2	3	4
$f(x)$	2	3	5	7

range. $f(3)=5$.

$\longrightarrow$

x	2	3	5	7
$f^{-1}(x)$	1	2	3	4.

$f^{-1}(5)=3$

As we have seen from the illustrations, the domain D and the range R are interchanged when we think of f^{-1}.

Hence,

$$\boxed{\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned}}$$

Our goal is to find the inverse function f^{-1}.

We can easily achieve the goal by following the guidelines:

STEP 1 : Verify that f is one-to-one function throughout its domain.

STEP 2 : 1) Solve the equation $y=f(x)$ for x in terms of y.

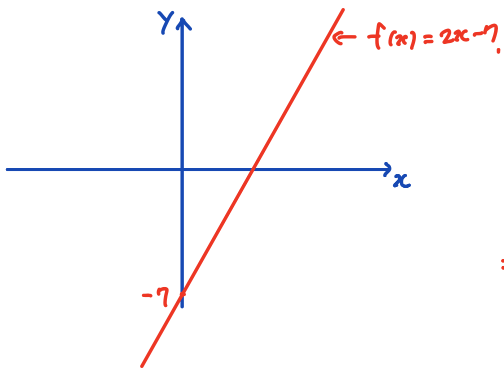
2) Interchange x and y. Then 3) replace y by $f^{-1}(x)$.

STEP 3 : Verify the following two conditions:

$f^{-1}(f(x)) = x$ for every x in the domain of f,

$f(f^{-1}(x)) = x$ for every x in the range of f.

Ex Let $f(x) = 2x - 7$. Find $f^{-1}(x)$.



: It is increasing

$\Rightarrow$ It is one-to-one.

$y = f(x) \longrightarrow y = 2x - 7$: Solve it for x.

$\quad +7 \qquad +7$

$\longrightarrow \dfrac{y+7}{2} = \dfrac{2x}{2}$

$\longrightarrow x = \dfrac{y+7}{2}$: interchange x and y

$\longrightarrow y = \dfrac{x+7}{2}$: Replace y by $f^{-1}(x)$

$\longrightarrow \boxed{f^{-1}(x) = \dfrac{x+7}{2}}$: It is the inverse function.

* Check!

$f(f^{-1}(x)) = f\left(\dfrac{x+7}{2}\right) = 2\cdot\left(\dfrac{x+7}{2}\right) - 7$

$= (x+7) - 7$

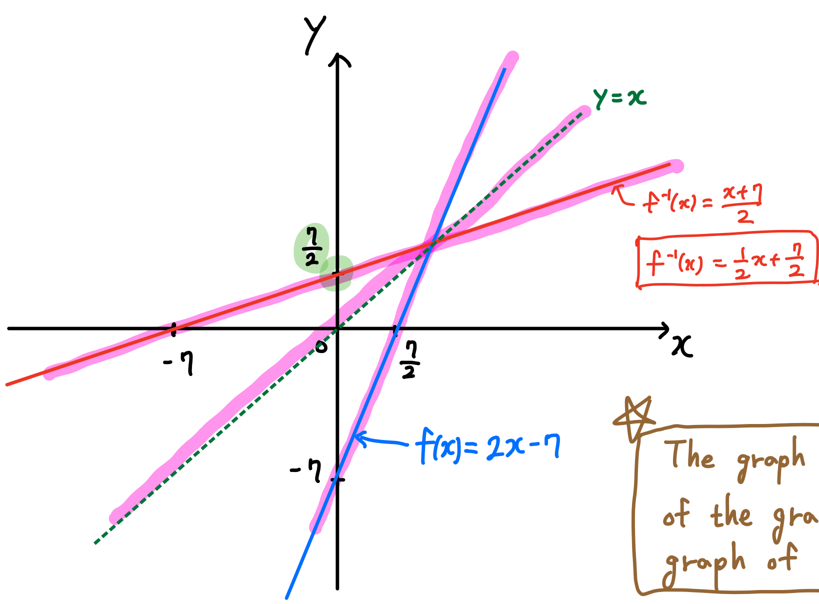
$= x$

* $f(x) = 2x - 7$

$f^{-1}(f(x)) = f^{-1}(2x-7) = \dfrac{(2x-7)+7}{2}$

$= \dfrac{2x}{2} = x$

* $f^{-1}(x) = \dfrac{x+7}{2}$



☆ The graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ through the graph of $y = x$!