Section 4.5. Continued.

* Recall

Guidelines for sketching the graph of a rational function.

Assume that $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials that have no common factor.

STEP 1. Find the x-intercept (solve $g(x) = 0$) and plot on the x-axis.

STEP 2. Find the zeros of $h(x)$: they will be the vertical asymptotes.

Then plot the vertical asymptotes with dashes.

STEP 3. Find the y-intercept (if it exists) and plot on the y-axis.

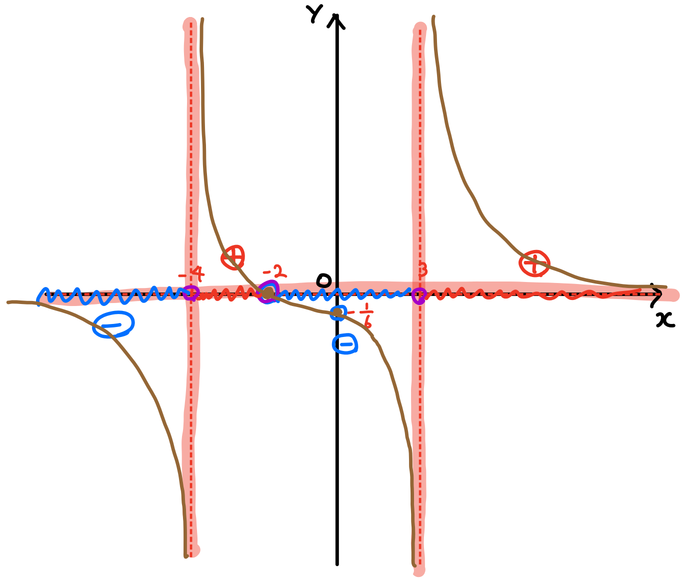
STEP 4. Find the horizontal asymptotes (or oblique asymptotes) and plot it with dashes.

STEP 5. Find the points where the graph of $f(x)$ and the horizontal asymptotes intersect.

STEP 6. Check whether $f(x)$ is above or below the x-axis or horizontal asymptotes (or oblique asymptotes), then sketch the graph of f.

Ex Sketch the graph of $f(x)=\dfrac{x+2}{x^2+x-12}$ $\left(\dfrac{g(x)}{h(x)}\right.$ where $\left.\deg(h)=\deg(g)+1\right)$

$$=\frac{(x+2)}{(x+4)(x-3)}$$



STEP 1. x-intercept.

$f(x)=\dfrac{x+2}{x^2+x-12}$

$y=\dfrac{x+2}{x^2+x-12}$ Set $y=0$ $\dfrac{x+2}{x^2+x-12}=0.$ $\Rightarrow x+2=0.$ $\Rightarrow \boxed{x=-2},$

$\left(\boxed{(x+4)(x-3)}\neq 0 \text{ when } x=-2.\right)$

STEP 2. When (the denominator) $=0$?

$x^2+x-12=0.$

$(x+4)(x-3)=0.$

$\downarrow$ Z.F.T.

$x+4=0$ or $x-3=0.$ $\longrightarrow$ $\boxed{x=-4 \text{ or } x=3}$ Vertical asymptotes

STEP 3. y-intercept.

$f(x)=\dfrac{x+2}{x^2+x-12}$

$y=\dfrac{x+2}{x^2+x-12}$ Set $x=0$ $\longrightarrow$ $y=\dfrac{0+2}{0+0-12}=\dfrac{2}{-12}=-\dfrac{1}{6}.$

$\Rightarrow \left(0,-\frac{1}{6}\right)$ is y-intercept.

STEP 4. Find the horizontal asymptote.

$f(x)=\dfrac{x+2}{x^2+x-12}$ $\leftarrow$ degree $=1$, $\leftarrow$ degree $=2$ $\Rightarrow$ According to the Theorem (see 10/7) $\boxed{x\text{-axis}}$ is the horizontal asymptote.

STEP 5. Do the graph of $f(x)$ and the horizontal asymptote intersect?

$f(x)=\dfrac{x+2}{x^2+x-12}$ x-axis, $\boxed{y=0}$

$\boxed{y=\dfrac{x+2}{x^2+x-12}.}$ $\longrightarrow$ Set $\dfrac{x+2}{x^2+x-12}=0.$: We solved it in STEP 1. $\Rightarrow \boxed{x=-2}.$

STEP 6. When the graph of $f(x)$ is above / below the horizontal asymptote? (or x-axis).

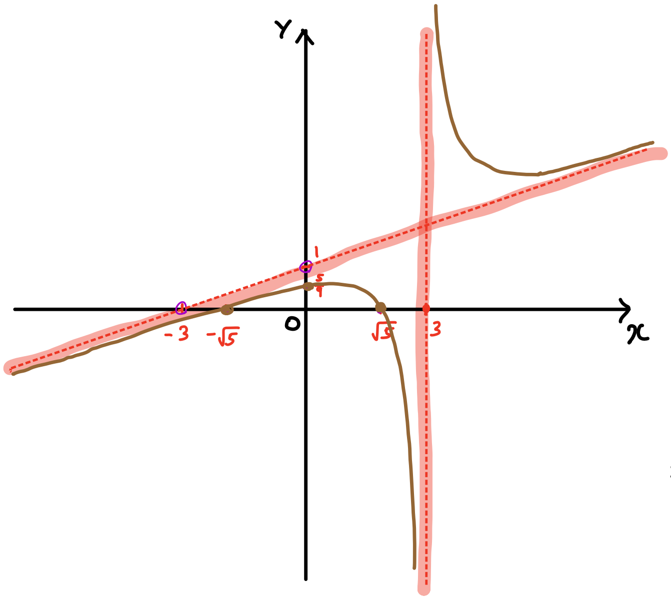
$$f(x)=\frac{x+2}{(x+4)(x-3)}$$

	$(-\infty,-4)$	-4	$(-4,-2)$	-2	$(-2,3)$	3	$(3,\infty)$
$f(x)=\dfrac{x+2}{(x+4)(x-3)}$	⊖		⊕		⊖		⊕.
$(x+2)$	$-$		$-$	0	$+$		$+$
$(x-3)$	$-$		$-$		$-$	0	$+$
$(x+4)$	$-$	0	$+$		$+$		$+$

$x+4=0$ if $x=-4$

$\underline{\text{Ex}}$ Sketch the graph of $f(x) = \dfrac{x^2-5}{3x-9}$ $\left(\dfrac{g(x)}{h(x)}\right.$ where $\left.\boxed{\deg(g)=\deg(h)+1}\right)$

In this case, we have oblique asymptote, instead of horizontal asymptote.



STEP 1. x-intercept.

$f(x) = \dfrac{x^2-5}{3x-9}$

$Y = \dfrac{x^2-5}{3x-9}$. $\xrightarrow{\text{Set } Y=0}$ $\dfrac{x^2-5}{3x-9} = 0$. $\Rightarrow x^2-5=0$. $\Rightarrow x^2 = 5$. $\to x = +\sqrt{5}$ or $x = -\sqrt{5}$. $\Rightarrow \boxed{(\sqrt{5},0)}$ and $\boxed{(-\sqrt{5},0)}$

STEP 2. When (the denominator) $= 0$?

$3x - 9 = 0$.

$3x = 9$

$\boxed{x = 3}$: Vertical Asymptote!

STEP 3. Y-intercept.

$f(x) = \dfrac{x^2-5}{3x-9}$

$Y = \dfrac{x^2-5}{3x-9}$. $\xrightarrow{\text{Set } x=0}$ $Y = \dfrac{0-5}{0-9} = \dfrac{-5}{-9} = \dfrac{5}{9}$.

$\Rightarrow \boxed{\left(0, \frac{5}{9}\right)}$

STEP 4. Find the ~~horizontal~~ oblique asymptote.

$f(x) = \dfrac{x^2-5}{3x-9}$

Long division : divide x^2-5 by $3x-9$.

$x^2+0\cdot x-5$

$$\begin{array}{r|l} & \boxed{\tfrac{1}{3}x+1} \\ 3x-9 & x^2+0\cdot x-5 \\ & \boxed{x^2-3x} \\ \hline & 3x-5 \\ & \boxed{3x-9} \\ \hline & \boxed{4} \end{array}$$

$\frac{1}{3}x(3x-9) = x^2-3x$

$f(x) = \dfrac{x^2-5}{3x-9} = \dfrac{(3x-9)(\frac{1}{3}x+1)+4}{3x-9}$

$= \dfrac{\cancel{(3x-9)}(\frac{1}{3}x+1)}{\cancel{(3x-9)}} + \dfrac{4}{3x-9}$

$= \boxed{\tfrac{1}{3}x+1} + \dfrac{4}{3x-9}$

it is our oblique asymptote!

$\boxed{Y = \tfrac{1}{3}x+1}$

STEP 5. Do the graph of $f(x)$ and the ~~horizontal~~ oblique asymptote intersect ?

$f(x) = \dfrac{x^2-5}{3x-9}$

$\boxed{Y = \dfrac{x^2-5}{3x-9}}$

$\boxed{Y = \tfrac{1}{3}x+1}$

$\dfrac{x^2-5}{3x-9} = \dfrac{1}{3}x + 1 \Rightarrow \cancel{\tfrac{1}{3}x+1} + \dfrac{4}{3x-9} = \cancel{\tfrac{1}{3}x+1}$

$\dfrac{4}{3x-9} = 0$: No solution

$\Rightarrow \boxed{\text{No intersection}}$

STEP 6. When the graph of $f(x)$ is above / below the ~~horizontal~~ oblique asymptote ?

: Use the information from STEP 1 ~ STEP 5 !