Section 4.5. Continued.

The following is the theorem on horizontal asymptotes.

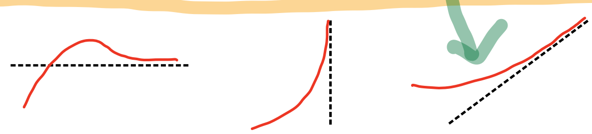
Let $f(x) = \dfrac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0}$, where $a_n \neq 0$ and $b_k \neq 0$.

(1) If $n < k$, then the x-axis is the horizontal asymptote.

Ex) $y = \frac{1}{x}$

(2) If $n = k$, then the line $y = \dfrac{a_n}{b_k}$ is the horizontal asymptote.

(3) If $n > k$, then there is no horizontal asymptote, but it may have oblique asymptote, if $n = k+1$.



Ex Find the horizontal asymptote for the graph of f, if it exists.

(a) $f(x) = \dfrac{4x^1 - 3}{7x^3 - 6x^2 + 1}$: $\boxed{x\text{-axis}}$ or $\boxed{y = 0}$

(b) $f(x) = \dfrac{4x^3 + 2x^2 + 3}{2x^3 - 3x + 1}$: $y = \frac{4}{2} \Rightarrow \boxed{y = 2}$

(c) $f(x) = \dfrac{4x^5 - 3x^2 + 4}{3x^3 - 4x + 2}$: No horizontal asymptote!

How to sketch the graph of rational function?

$\Downarrow$

Guidelines for sketching the graph of a rational function.

Assume that $\boxed{f(x) = \dfrac{g(x)}{h(x)}}$, where $g(x)$ and $h(x)$ are polynomials that have no common factor.

$y = \dfrac{g(x)}{h(x)}$, $\quad 0 = \dfrac{g(x)}{h(x)}$

STEP 1. Find the x-intercept (solve $g(x) = 0$) and plot on the x-axis.

STEP 2. Find the zeros of $h(x)$: they will be the vertical asymptotes.
(solve $h(x) = 0$).
Then plot the vertical asymptotes with dashes.

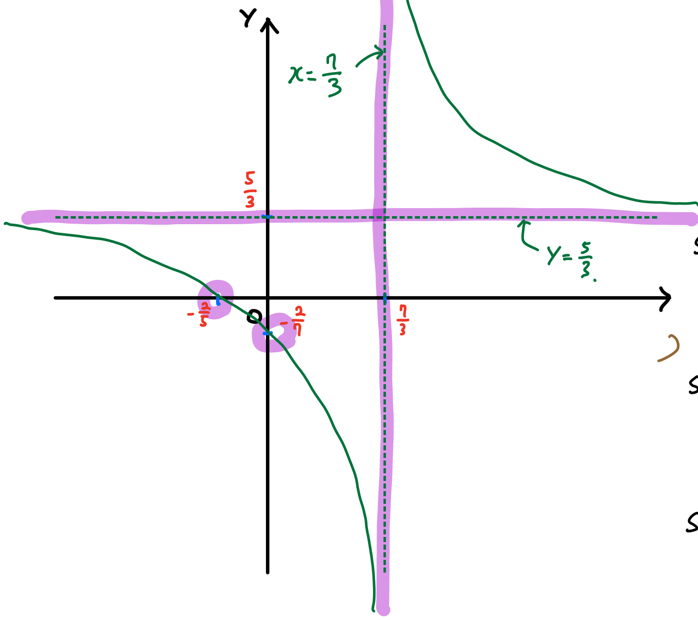
STEP 3. Find the y-intercept (if it exists) and plot on the y-axis.
$y = \dfrac{g(x)}{h(x)} \rightarrow$ replace x by 0.

STEP 4. Find the horizontal asymptotes and plot it with dashes.
(or oblique asymptotes)

STEP 5. Find the points where the graph of $f(x)$ and the horizontal asymptotes intersect.

STEP 6. Check whether $f(x)$ is above or below the x-axis or horizontal asymptotes (or oblique asymptotes), then sketch the graph of f.

Ex Sketch the graph of $f(x) = \dfrac{5x+2}{3x-7}$ $\left(\dfrac{g(x)}{h(x)} \text{ where } \deg(g) = \deg(h) \right)$



STEP 1. x-intercept.

Solve $5x+2=0$. $\Rightarrow x = -\frac{2}{5}$

$\Rightarrow \boxed{\left(-\frac{2}{5}, 0\right)}$

STEP 2. When (the denominator) = 0 ?

Solve $3x-7=0 \Rightarrow \boxed{x = \frac{7}{3}}$

Vertical asymptote!

STEP 3. y-intercept.

$f(x) = \dfrac{5x+2}{3x-7}$ → Set $x=0$

$y = \dfrac{5x+2}{3x-7}$: $y = \dfrac{0+2}{0-7} = -\frac{2}{7} \Rightarrow \boxed{\left(0, -\frac{2}{7}\right)}$

STEP 4. Find the horizontal asymptote.

$f(x) = \dfrac{5x+2}{3x-7}$: $5x+2$ and $3x-7$ have the same degree

$\Rightarrow \boxed{y = \frac{5}{3}}$ is a horizontal asymptote.

STEP 5. Do the graph of $f(x)$ and the horizontal asymptote intersect ?

$f(x) = \dfrac{5x+2}{3x-7}$ $\qquad y = \frac{5}{3}$

$\Downarrow$

$y = \dfrac{5x+2}{3x-7}$ Solve $\dfrac{5x+2}{3x-7} = \dfrac{5}{3}$, $\xrightarrow{\times 3\cdot(3x-7)}$ $3\cdot(5x+2) = 5(3x-7) \Rightarrow \cancel{15x}+6 = \cancel{15x}-35$, $6=-35$

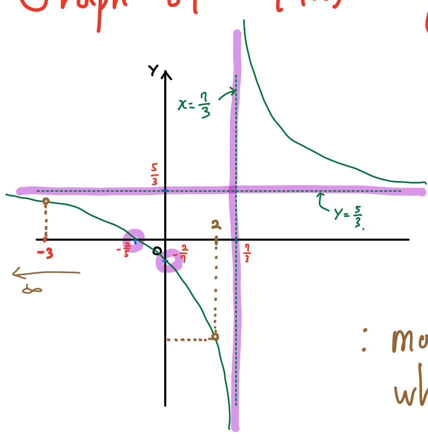
$\Rightarrow$ No solution.

$\Rightarrow$ No intersection!

STEP 6. When the graph of $f(x)$ is above / below the horizontal asymptote ?

: Use the information from STEP 1 ~ STEP 5 !

* Graph of $f(x) = \dfrac{(5x+2)(x-2)(x+3)}{(3x-7)(x-2)(x+3)}$



↳ if $x = 2$ or -3, the function is not defined.

if $x \neq 2$ and -3, then $x-2 \neq 0$, $x+3 \neq 0$.

$\Rightarrow f(x) = \dfrac{(5x+2)(x-2)\cancel{(x+3)}}{(3x-7)\cancel{(x-2)}\cancel{(x+3)}} = \dfrac{5x+2}{3x-7}$

: make two holes at the points on the graph whose x-coordinate is 2 or -3.

Can we do the reverse?

Find an equation of a rational function f that satisfies the following conditions:

$x - (-\frac{2}{5}) = (x + \frac{2}{5})$ is a factor of $g(x)$.

x-intercept: $-\frac{2}{5}$, vertical asymptote: $x = \frac{7}{3}$,

$(x - \frac{7}{3})$ is a factor of $h(x)$.

horizontal asymptote: $y = \frac{5}{3}$, and a hole at $x = 2$.

leading coefficient of $g(x)$

leading coefficient of $h(x)$.

$$f(x) = \frac{g(x)}{h(x)}. \quad \frac{5\,(x+\frac{2}{5})(x-2)}{3\,(x-\frac{7}{3})(x-2)}$$

$(x-2)$ is the common factor of $g(x)$ and $h(x)$.

$$\Downarrow$$

$$\boxed{f(x) = \frac{(5x+2)(x-2)}{(3x-7)(x-2)}}$$