Section 4.4. ~~Complex and~~ Rational Zeros of Polynomials.

$\cancel{i,\ 3+2i,\ \ldots}$

(We will come back to complex zeros at the end of the semester if time permites.)

Q : Why we care about rational zeros ?

A : If we know a rational zero, we can factor a given polynomial ! (Long division, Synthetic division ...)

Ex If we know that $\frac{2}{5}$ is a zero of $f(x) = 10x^3+11x^2-x-2$, then

$\Updownarrow$

$(x-\frac{2}{5})$ is a factor.

$10x^3+11x^2-x-2$

$\Downarrow$

$$\begin{array}{c|cccc}
\frac{2}{5} & 10 & 11 & -1 & -2 \\
 & \downarrow & 4 & 6 & 2 \\
\hline
 & 10 & 15 & 5 & \boxed{0}
\end{array}$$

remainder.

$10x^2+15x+5$: quotient

$$10x^3+11x^2-x-2 = \left(x-\frac{2}{5}\right)\cdot\left(10x^2+15x+5\right)$$

Q. Given a polynomial $f(x)$, how can we find a rational zero of it?

A. Use the following theorem!

Theorem on Rational Zeros of a Polynomial

If the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has an integer coefficients and $\frac{c}{d}$ is a rational zero of $f(x)$ such that c and d have no common prime factor, then

(1) the numerator c of the zero is a factor of the constant term a_0

(2) the denominator d of the zero is a factor of the leading coefficient a_n

The above theorem implies the following:

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has a rational zero, then it is of the following form.

$$\boxed{x = \frac{\text{a factor of the constant term } a_0}{\text{a factor of the leading coefficient } a_n}}$$

Ex Factor $4x^2 - 8x + 3$

leading coefficient (4) constant (3).

Let $x = \frac{c}{d}$ is a zero. $\Rightarrow$ c is a factor of 3 : $-1, 1, -3, 3$

$\Rightarrow$ d is a factor of 4 : $-1, 1, -2, 2, -4, 4$.

$$x = \frac{\text{one of } \{-1, 1, -3, 3\}}{\text{one of } \{-1, 1, -2, 2, -4, 4\}}$$

Let's try $x = \frac{3}{2}$. $\Rightarrow$ $4\cdot\left(\frac{3}{2}\right)^2 - 8\cdot\left(\frac{3}{2}\right) + 3 = \cancel{4}^{1}\cdot\frac{9}{\cancel{4}_1} - \cancel{8}^{4}\cdot\frac{3}{\cancel{2}_1} + 3$

$= 9 - 12 + 3 = \underline{0}$

Hence $\underline{x = \frac{3}{2} \text{ is the zero!}}$

$\hookrightarrow f(x)$ is factored by $\left(x - \frac{3}{2}\right)$

$$\begin{array}{c|ccc} \frac{3}{2} & 4 & -8 & 3 \\ & & 6 & -3 \\ \hline & 4 & -2 & \boxed{0} \end{array}$$

$\downarrow$

$4x - 2$

$4x^2 - 8x + 3 = \left(x - \frac{3}{2}\right)(4x - 2)$

$= \left(x - \frac{3}{2}\right)\cdot 2(2x - 1)$

$= \boxed{(2x-3)(2x-1)}$

Ex Show that $2x^3+6x^2-1$ has no rational zeros.

Suppose there is a rational zero : $x = \frac{c}{d}$.

According to the theorem, c is a $\underline{\text{factor of } -1}$, $\to \{1, -1\}$

d is a $\underline{\text{factor of } 2}$. $\to \{1, -1, 2, -2\}$

Then $\frac{c}{d}$ can be $\frac{1}{1}, \frac{1}{-1}, \frac{1}{2}, \frac{1}{-2}, \frac{-1}{1}, \frac{-1}{-1}, \frac{-1}{2}, \frac{-1}{-2}$

$\downarrow$ $1, -1, \frac{1}{2}, -\frac{1}{2}, -1, 1, -\frac{1}{2}, \frac{1}{2}$

$\underbrace{\qquad}$ $1, -1, \frac{1}{2}$, and $-\frac{1}{2}$.

$2x^3+6x^2-1$

$1 : 2\cdot1^3+6\cdot1^2-1 = 2+6-1 = 7 \neq 0$

$-1 : 2\cdot(-1)^3+6\cdot(-1)^2-1 = 2\cdot(-1)+6\cdot1-1 = -2+6-1 = 3 \neq 0.$

$\frac{1}{2} : 2\cdot\left(\frac{1}{2}\right)^3+6\cdot\left(\frac{1}{2}\right)^2-1 = 2\cdot\frac{1}{8}+6\cdot\frac{1}{4}-1 = \frac{1}{4}+\frac{6}{4}-1 = \frac{7}{4}-1 = \frac{3}{4} \neq 0.$

$-\frac{1}{2} : 2\cdot\left(-\frac{1}{2}\right)^3+6\cdot\left(-\frac{1}{2}\right)^2-1 = 2\cdot\left(-\frac{1}{8}\right)+6\cdot\frac{1}{4}-1 = -\frac{1}{4}+\frac{6}{4}-1 = \frac{5}{4}-1 = \frac{1}{4} \neq 0.$

None of them are zero.

Hence, we can conclude that the given polynomial has no rational zero.

Ex Find all rational solutions of the equation

$$15x^4 - x^3 - 17x^2 + x + 2$$

$x = \frac{c}{d}$ $\xrightarrow{\text{by the Theorem}}$ c is a factor of 2. $\to \{1, -1, 2, -2\}$.

d is a factor of 15. $\to \{1, -1, 3, -3, 5, -5, 15, -15\}$

Let us try $c = 1$, $d = 1$, $\Rightarrow x = \frac{1}{1} = 1$.

$x = 1$: $15 \cdot 1^4 - 1^3 - 17 \cdot 1^2 + 1 + 2 = 15 - 1 - 17 + 1 + 2 = 0$.

$x = 1$ is a zero.

$\Downarrow$

$(x-1)$ is a factor!

$$\begin{array}{c|ccccc} 1 & 15 & -1 & -17 & 1 & 2 \\ & & 15 & 14 & -3 & -2 \\ \hline & 15 & 14 & -3 & -2 & \boxed{0} \end{array}$$

$\downarrow$

$15x^3 + 14x^2 - 3x - 2$

$15x^4 - x^3 - 17x^2 + x + 2$

$= (x-1)\left(\boxed{15x^3 + 14x^2 - 3x - 2}\right)$

$= (x-1)(x+1)(15x^2 - x - 2) = (x-1)(x+1)(x - \frac{2}{5})$

$(15x+5)$

$x = \frac{s}{t}$

Let $s = 2$, $t = 5$ $\to$ $x = \frac{2}{5}$.

$15x^3 + 14x^2 - 3x - 2 = (x+1)(15x^2 - x - 2)$

If $x = \frac{e}{f}$ is a zero, then e is a factor of -2,

f is a factor of 15.

Let us try $e = -1$ and $f = 1$, $\Rightarrow$ $x = \frac{-1}{1} = -1$.

$x = -1$: $15 \cdot (-1)^3 + 14 \cdot (-1)^2 - 3 \cdot (-1) - 2 = -15 + 14 + 3 - 2 = 0$.

$x = -1$ is a zero

$\Downarrow$

$(x+1)$ is a factor.

$$\begin{array}{r|rrrr} -1 & 15 & 14 & -3 & -2 \\ & & -15 & 1 & 2 \\ \hline & 15 & -1 & -2 & \boxed{0} \end{array}$$

$\underbrace{15 \quad -1 \quad -2}_{15x^2-x-2}$

$\Rightarrow 15x^3+14x^2-3x-2$
$= (x+1)(15x^2-x-2)$

Thus $15x^4-x^3-17x^2+x+2 = (x-1)(15x^3+14x^2-3x-2)$
$= (x-1)(x+1)(15x^2-x-2)$

$15x^2-x-2$

If $x=\frac{s}{t}$ is a zero, then s is a factor of -2,
t is a factor of 15.

Let us try $s=2$ and $t=5$. $\Rightarrow x=\frac{2}{5}$.

$x=\frac{2}{5}$: $15\cdot\left(\frac{2}{5}\right)^2-\frac{2}{5}-2 = \cancel{15}^{3}\cdot\frac{4}{\cancel{25}_{5}}-\frac{2}{5}-2 = \frac{12}{5}-\frac{2}{5}-2 = 0$

$x=\frac{2}{5}$ is a zero

$\downarrow$

$x-\frac{2}{5}$ is a factor.

$$\begin{array}{r|rrr} \frac{2}{5} & 15 & -1 & -2 \\ & & 6 & 2 \\ \hline & 15 & 5 & \boxed{0} \end{array}$$

$\underbrace{15 \quad 5}_{15x+5}$

Hence, $15x^4-x^3-17x^2+x+2 = \underbrace{(x-1)}_{=0}\underbrace{(x+1)}_{=0}\underbrace{\left(x-\frac{2}{5}\right)}_{=0}\underbrace{(15x+5)}_{=0}$

By the Zero Factor Theorem, $\boxed{x=1,-1,\frac{2}{5},-\frac{1}{3}}$ are all rational solutions