HW 6 : due this Sunday (10/10) at 11:59 pm.

Section 4.3 Zeros of Polynomials.

Given a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$,

$x = s$ is the solution of $f(x)$ if $f(s) = a_n s^n + a_n s^{n-1} + \cdots + a_1 s + a_0 = 0$

What can we say about the zeros of the polynomial $f(x)$?

Ex

1) $f(x) = 2x + 3 = 0$, $2x = -3$, $\boxed{x = -\frac{3}{2}}$ 1 zero / degree : 1

2) $f(x) = 3x^2 - 15 = 0$, $3(x^2 - 5) = 0$, $3(x^2 - (\sqrt{5})^2) = 0$, $3(x + \sqrt{5})(x - \sqrt{5}) = 0$ → $\boxed{x = \sqrt{5}, -\sqrt{5}}$ 2 zeros / degree : 2

3) $f(x) = x^2 + 6x + 9 = 0$ $\quad x^2 + 2 \cdot x \cdot 3 + 3^2 = 0 \Rightarrow (x+3)^2 = 0$

use $x^2 + 2xy + y^2 = (x+y)^2$

$(x+3) \cdot (x+3) = 0 \xrightarrow{\text{Z.F.T.}} x+3 = 0$ or $x+3 = 0$, $\boxed{x = -3 \text{ or } x = -3}$

1 zero of $\boxed{\text{multiplicity 2}}$ degree 2

4) $f(x) = x^3 - x^2 + x - 1 = 0$ $\quad (x^3 + x) + (-x^2 - 1) = 0$

$x(x^2+1) - (x^2+1) = 0$

$(x^2+1)(x-1) = 0 \xrightarrow{\text{Z.F.T.}} x^2 + 1 = 0$ or $x - 1 = 0$

No such x $\quad \boxed{x = 1}$

1 zero / degree 3

Observation : The number of zeros does not exceed the degree of the polynomial.

Theorem If the degree of $f(x)$ is n, $f(x) = 0$ has $\boxed{\text{at most } n}$ different real zeros.

If $f(x)$ has n different real zeros $c_1, c_2, \cdots, c_n$, then the factored form of $f(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$

leading coefficient of $f(x)$.

Ex Find a polynomial $f(x)$ in factored form that has degree 4;

has zeros, 3, -2, 0, -1 ; and satisfies $f(1) = -16$.

degree : 4

zeros : $\underbrace{3, -2, 0, -1}_{\text{all different.}}$

By the theorem, $f(x) = a(x-3)(x-(-2))(x-0)(x-(-1))$

$f(x) = a(x-3)(x+2)\,x\,(x+1)$ $\rightarrow$ $\boxed{f(x) = \frac{4}{3}(x-3)(x+2)\,x\,(x+1)}$

We know that $f(1) = -16$.

However, $f(1) = a\cdot(1-3)\cdot(1+2)\cdot 1\cdot(1+1)$

$= a\cdot(-2)\cdot 3\cdot 1\cdot 2$

$= -12a$

$-16 = -12a$

$\downarrow \div(-12)$

$a = \frac{-16}{-12} = \frac{4}{3}$

We already have seen several polynomials whose factored form contains $(x-c)^2$

Ex $x^2 + 6x + 9 = (x+3)^2 = (x-(-3))^2$.

In general, if $f(x)$ contains $\boxed{(x-c)^m}$ in its factored form,

we say that c is a zero of $\boxed{\text{multiplicity } m}$

or root of multiplicity m

If $f(x)$ has a zero with a zero of multiplicity > 1, graphing the graph of $y = f(x)$ requires the following facts.

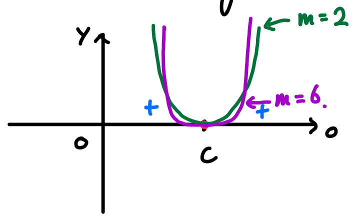
$f(x) = (x-c)^m \times$ (some polynomial)

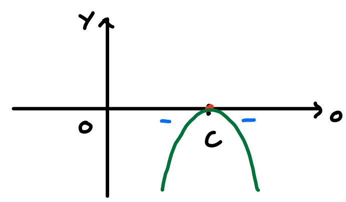
$y = (x-c)^m \times$ (some polynomial) $\to (c, 0)$ is on the graph

If $(x-c)^m$ is a factor of $f(x)$, then the graph of $f(x)$ near $(c, 0)$ shows the following local behavior:

1) If m is even

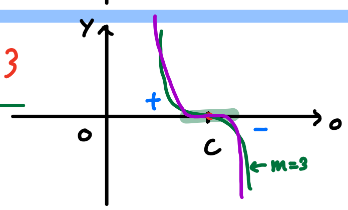
∴ ① Flat at $(c, 0)$
② Sign unchanged

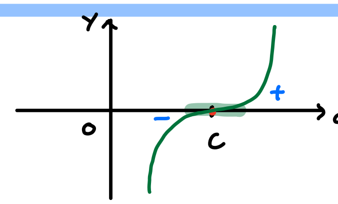




2) If m is odd ≥ 3

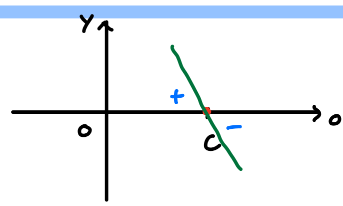
∴ ① Flat at $(c, 0)$
② Sign changed

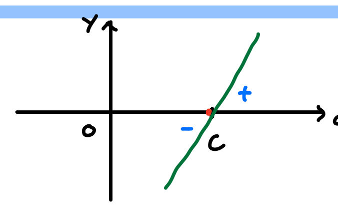




3) If m is 1

∴ ① Sharp at $(c, 0)$
② Sign changed





In all cases, as m increase, the graph becomes flatter near $(c, 0)$

Keeping it in your mind, let's see some example.

Ex Find zeros of the polynomial $f(x) = 2(x-3)^2(x+2)^3(x-1)$.

y

$x-(-2)$

State the multiplicity of each, and then sketch the graph of f.

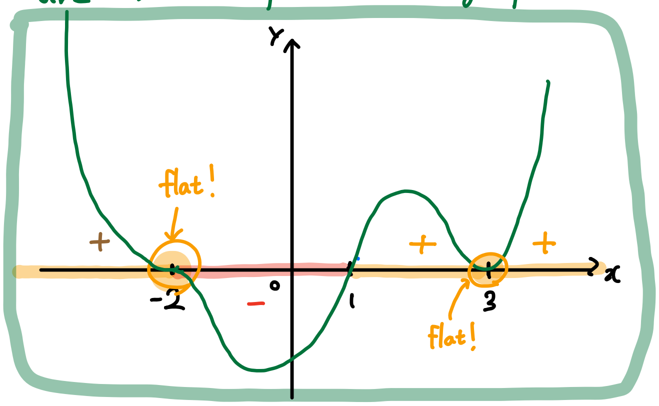
zeros : 3 with multiplicity 2

-2 with multiplicty 3

1 with multiplicty 1

$\Rightarrow$ $(3,0)$, $(-2,0)$, and $(1,0)$ are x-intercept of the graph

	$(-\infty,-2)$	-2	$(-2,1)$	1	$(1,3)$	3	$(3,\infty)$
$f(x)=2(x-3)^2(x+2)^3(x-1)$	+		−		+		+
$(x-3)^2$	+		+		+	0	+
$(x+2)^3$	−	0	+		+		+
$(x-1)$	−		−	0	+		+



The theorem that we have seen earlier is still true if we count the zeros with multiplicity :

Theorem If the degree of $f(x)$ is n, $f(x)=0$ has at most n real zeros, where we count every zero with its multiplicity.

Ex Express $f(x) = x^5 + x^4 - 7x^3 - 7x^2$ as a product of linear factors, and find the five zeros of $f(x)$.

$$\begin{aligned} f(x) &= x^5 + x^4 - 7x^3 - 7x^2 \\ &= x^2(x^3 + x^2 - 7x - 7) \\ &= x^2\big((x^3+x^2) + (-7x-7)\big) \\ &= x^2\big(x^2(x+1) - 7(x+1)\big) \\ &= x^2(x+1)(x^2-7) \\ &= x^2(x+1)(x^2-(\sqrt{7})^2) \\ &= x^2(x+1)(x+\sqrt{7})(x-\sqrt{7}). \end{aligned}$$

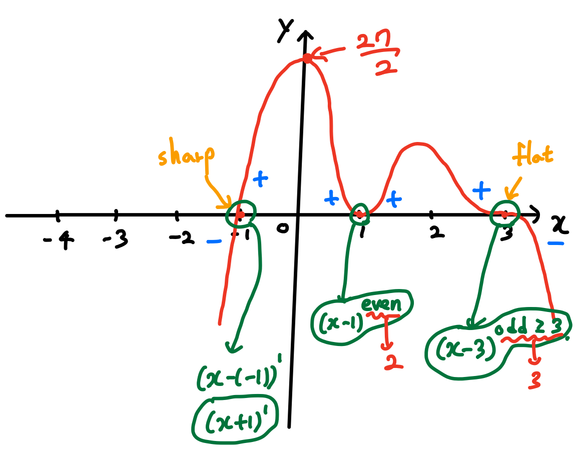
Hence, the five zeros of $f(x)$ are $\boxed{0,\ 0,\ -1,\ -\sqrt{7},\ \text{and } \sqrt{7}}$

0 is a zero of multiplicity 2.

Ex Shown in the below picture are all zeros of a polynomial function f.

(a) Find a factored form for f that has minimal degree.

(b) Assuming the leading coefficient of f is $-\frac{1}{2}$, find the y-intercept.



(a) $\boxed{a \cdot (x+1)(x-1)^2(x-3)^3}$

(b) $f(x) = -\frac{1}{2}(x+1)(x-1)^2(x-3)^3$.

set $x=0$

$$-\frac{1}{2}(0+1)(0-1)^2(0-3)^3 = \boxed{\frac{27}{2}}$$