HW5: due tomorrow at 11:59 pm
HW6: will be posted today

Division Algorithm for Polynomials.

If $f(x)$ and $p(x)$ are polynomials and if $p(x) \neq 0$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = p(x) \cdot q(x) + r(x)$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $p(x)$.

Now, we will focus on the special case when $p(x) = x - c$.

For any polynomial $f(x)$, by the division algorithm, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = (x-c) \cdot q(x) + r(x)$$

where either $r(x) = 0$ or $\deg(r(x)) < \boxed{\deg(x-c)}$. $= 1$

Since $\deg(x-c) =$ 1, $\deg(r(x)) =$ 0.

Thus, $r(x) = d$ for some real number d.

$$\Rightarrow \boxed{f(x) = (x-c) \cdot q(x) + d} \quad = f(c)$$

c c c

Set $x = c$, then we have

$$f(c) = (c-c)\cdot q(c) + d$$

$$f(c) = 0 \cdot q(c) + d$$

$$\boxed{f(c)} = 0 + d = \underline{\underline{d}} \Rightarrow d = f(c)$$

Hence, $f(x) = (x-c)\cdot q(x) + \boxed{f(c)}$

We just proved the following theorem:

Remainder Theorem

If a polynomial $\boxed{f(x)}$ is divided by $\boxed{x-c}$, then the remainder is $\boxed{f(c)}$.

-1

$x-(-1) = x+1$

-1

$f(-1)$

Ex If $f(x) = 2x^3 - 4x^2 + x + 5$, use the remainder theorem to find $f(-1)$.

Proof 1) I will divide $f(x)$ by $x-(-1) = x+1$.

Then, the remainder Theorem says the remainder is $f(-1)$.

$$\begin{array}{r}
2x^2 - 6x + 7 \\
x+1 \overline{\big) 2x^3 - 4x^2 + x + 5} \\
2x^3 + 2x^2 \qquad\qquad \\
\hline
-6x^2 + x \qquad \\
-6x^2 - 6x \qquad \\
\hline
7x + 5 \\
7x + 7 \\
\hline
-2
\end{array}$$

$-2 = f(-1)$.

$\boxed{f(-1) = -2}$

Proof 2)

$f(x) = 2x^3 - 4x^2 + x + 5$.

$f(-1) = 2\cdot(-1)^3 - 4\cdot(-1)^2 + (-1) + 5$

$= 2\cdot(-1) - 4\cdot 1 + (-1) + 5$

$= -2 - 4 - 1 + 5 = \boxed{-2}$,

Factor Theorem

A polynomial $f(x)$ has factor $x-c$ if and only if $\boxed{f(c)=0}$.

$(x-c)$ divides $f(x)$

$\updownarrow$

$f(x)=(x-c)\cdot g(x)$ for some polynomial $g(x)$.

$\updownarrow$

$f(x)=(x-c)\cdot g(x)+0$ $= f(c)$.

Ex Show that $x-3$ is the factor of $f(x)=2x^3-5x^2-4x+3$.

Proof 1)

$$\begin{array}{r}
2x^2+x-1 \\
x-3\,\overline{)\,2x^3-5x^2-4x+3} \\
\underline{2x^3-6x^2} \\
x^2-4x \\
\underline{x^2-3x} \\
-x+3 \\
\underline{-x+3} \\
0
\end{array}$$

$f(x)=2x^3-5x^2-4x+3=(x-3)(2x^2+x-1)$

$\Rightarrow$ $(x-3)$ is a factor.

Proof 2) ⟨Factor Theorem⟩,

$f(x)$ has factor $x-3$ if $f(3)=0$.

$f(3)=2\cdot 3^3-5\cdot 3^2-4\cdot 3+3$

$=2\cdot 27-5\cdot 9-12+3$

$=54-45-12+3=0$.

By the factor theorem, $(x-3)$ divides $f(x)$.

Ex Find a polynomial $f(x)$ of degree 4 that has 0, 3, 1, and -2 as zeros such that $\boxed{f(-1)=16}$.

$(x-0)=x$ divides $f(x)$,

$(x-3)$ divides $f(x)$,

$(x-1)$ divides $f(x)$,

if -2 is a zero of $f(x)$, then it means $f(-2)=0$.

Factor theorem says $f(-2)=0$ implies "$(x-(-2))$ divides $f(x)$".

$(x-(-2)) = (x+2)$

$\Rightarrow x,\ x-3,\ x-1$, and $x+2$ divide $f(x)$

$\Rightarrow \boxed{x\cdot(x-3)(x-1)(x+2)}$ divides $f(x)$. $\quad \boxed{f(x)=-2x(x-3)(x-1)(x+2)}$

degree? $\boxed{4}$ $\quad$ it also has degree 4.

$\boxed{f(-1)=a\cdot(-1)\cdot(-1-3)(-1-1)(-1+2)}$ $\quad -8a=16,\ \boxed{a=-2}$

$\Rightarrow f(x)=a\cdot x(x-3)(x-1)(x+2)$, but $\boxed{f(-1)=16}$

$a=-2$ some real number. $\quad a\cdot(-1)\cdot(-4)\cdot(-2)\cdot 1=16$

When two polynomials $f(x)$ and $p(x)\neq 0$ are given,

we know how to divide $f(x)$ by $p(x)$: Long division.

When $p(x)=x-c$, there is an easier way to do the division!

: Synthetic division.

Ex Divide $3x^4-5x^3-2x+7$ by $x-2$. $\quad (x-c),\ c=2$

$\boxed{\begin{aligned}&3x^4-5x^3-2x+7\\&=(x-2)(3x^3+x^2+2x+2)+11\end{aligned}}$

Proof 1). Long division :

$$\begin{array}{r}
3x^3+x^2+2x+2\\
x-2\,\big)\,3x^4-5x^3+0\cdot x^2-2x+7\\
3x^4-6x^3\qquad\qquad\qquad\\
\hline
x^3+0\cdot x^2\qquad\quad\\
x^3-2x^2\qquad\quad\\
\hline
2x^2-2x\quad\\
2x^2-4x\quad\\
\hline
2x+7\\
2x-4\\
\hline
11
\end{array}$$

Proof 2) $3x^4-5x^3+0x^2-2x+7$

$$\begin{array}{c|ccccc}
2 & 3 & -5 & 0 & -2 & 7\\
 & & 6 & 2 & 4 & 4\\
\hline
 & 3 & 1 & 2 & 2 & 11
\end{array}$$

$11 \leftarrow$ remainder

$\boxed{3x^3+x^2+2x+2} \leftarrow$ quotient.

Ex If $f(x)=2x^5-4x^3+3x^2-5$, use synthetic division to

find $f(3)$.

I will divide $f(x)$ by $(x-3)$. $\quad c$

The remainder theorem says that the remainder is $f(3)$,

$$2x^5-4x^3+3x^2-5 \longrightarrow 2x^5+0\cdot x^4-4\cdot x^3+3\cdot x^2+0\cdot x-5$$

$$\begin{array}{c|cccccc}
3 & 2 & 0 & -4 & 3 & 0 & -5 \\
 & & 6 & 18 & 42 & 135 & 405 \\ \hline
 & 2 & 6 & 14 & 45 & 135 & 400
\end{array}$$

$\underbrace{2x^4+6x^3+14x^2+45x+135}_{\text{quotient!}}$

$$2x^5-4x^3+3x^2-5=(x-3)(2x^4+6x^3+14x^2+45x+135)+\boxed{\underbrace{400}_{=f(3)}}$$

Ex Show that 7 is a zero of the polynomial

$$f(x)=x^3+x^2-65x+63.$$

7 is a zero of $f(x)$

$\Updownarrow$

$f(7)=0$

$\Updownarrow$ ← factor theorem.

$(x-7)$ divides $f(x)$

$x^3+x^2-65x+63$

$$\begin{array}{c|ccc|c}
7 & 1 & 1 & -65 & 63 \\
 & & 7 & 56 & -63 \\ \hline
 & 1 & 8 & -9 & 0
\end{array}$$

↑ remainder.

x^2+8x-9 : quotient.

$\Rightarrow f(x)=x^3+x^2-65x+63=(x-7)(x^2+8x-9).$

Hence, 7 is a zero of the polynomial $f(x)=x^3+x^2-65x+63$.