Section 3.7. Operations on Functions.

Given two real numbers, we get a new real number.
Ex 2, 7. $\rightarrow$ $2-7=-5$.

We know four operations $(+, -, \times, \div)$ on Real Numbers.

We will see five operations $(+, -, \times, \div, \circ)$ on Functions.

Composition!

Given two functions, we get a new function.

Sum, Difference, Product, and Quotient of Functions:

Given functions f and g,

$f+g$ (sum)

$f-g$ (difference)

fg (product)

$\frac{f}{g}$ (quotient)

are new functions, and they are defined as follows:

input, real numbers

$$
\begin{cases}
(f+g)(x) = f(x)+g(x) \\
(f-g)(x) = f(x)-g(x) \\
(fg)(x) = f(x)g(x) \\
\left(\frac{f}{g}\right)(x) = \left(\frac{f(x)}{g(x)}\right) \text{ when } \boxed{g(x)\neq 0}
\end{cases}
$$

$(f+g)(x)=f(x)+g(x)$: True.

$\cancel{f(x+y)=f(x)+f(y)}$: Not True.

$f(3) = 4\cdot 3 - 1 = 12 - 1 = 11$ $\qquad$ $g(3) = 2\cdot 3^2 = 2\cdot 9 = 18$.

Ex $f(x) = 4x - 1$, $g(x) = 2x^2$.

$\boxed{(f+g)(x) = f(x) + g(x)} = \underline{4x - 1 + 2x^2}$ $\qquad$ $(f+g)(3) =$

Proof 1) We know that $\boxed{(f+g)(x) = 4x - 1 + 2x^2}$

$(f+g)(3) = 4\cdot 3 - 1 + 2\cdot 3^2$
$= 12 - 1 + 18$
$= 11 + 18$
$= 29$.

Proof 2) $(f+g)(3) = f(3) + g(3) = 11 + 18 = \underline{29}$

$(f-g)(x) = f(x) - g(x) = \underline{4x - 1 - 2x^2}$

$(fg)(x) = f(x)g(x) = \underline{(4x-1)\cdot 2x^2}$ $\qquad$ $(f-g)(3) = f(3) - g(3) = 11 - 18 = \underline{-7}$.

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \underline{\frac{4x-1}{2x^2}}$ $\qquad$ $(fg)(3) = f(3)\cdot g(3) = 11\cdot 18 = \underline{198}$.

$\left(\frac{f}{g}\right)(3) = \frac{f(3)}{g(3)} = \underline{\frac{11}{18}}$

What can we say about the domains of these new functions?

Domain of $f+g$, $f-g$, and fg is $\underbrace{D(f)}_{\text{Domain of } f} \cap \underbrace{D(g)}_{\text{Domain of } g}$.

Domain of $\frac{f}{g}$ is $\boxed{D(f) \cap D(g)} - \{\underline{\text{the set of all } x \text{ such that } g(x) = 0}\}$.

$1 - x^2 \geq 0 \xrightarrow[(-1)]{\text{multiply}} x^2 - 1^2 \leq 0$

$(x+1)(x-1) \leq 0$ $\quad$ $[-1, 1]$

$\Rightarrow \boxed{-1 \leq x \leq 1} : D(f)$

$2x - 1 \geq 0$

$2x \geq 1$

$\boxed{x \geq \frac{1}{2}} \Rightarrow D(g) = [\frac{1}{2}, \infty)$

Ex $f(x) = \sqrt{1 - x^2}$ and $g(x) = \sqrt{2x - 1}$

$(f+g)(x) = f(x) + g(x) = \sqrt{1 - x^2} + \sqrt{2x - 1}$, Domain $= [-1, 1] \cap [\frac{1}{2}, \infty) = \underline{[\frac{1}{2}, 1]}$

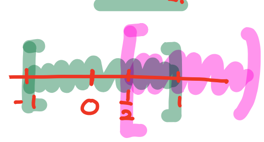
If we set $x = 0$, $\sqrt{1 - 0^2} = \sqrt{1}$ $\quad$ $\sqrt{2\cdot 0 - 1} = \sqrt{-1}$ Does not make sense!

$(f-g)(x) = f(x) - g(x) = \sqrt{1 - x^2} - \sqrt{2x - 1}$, Domain $= [-1, 1] \cap [\frac{1}{2}, \infty) = \underline{[\frac{1}{2}, 1]}$

$(fg)(x) = f(x)g(x) = \sqrt{1 - x^2}\cdot\sqrt{2x - 1}$, Domain $= [-1, 1] \cap [\frac{1}{2}, \infty) = \underline{[\frac{1}{2}, 1]}$

$\left(\frac{f}{g}\right)(\frac{1}{2}) = \frac{f(\frac{1}{2})}{g(\frac{1}{2})} = \frac{\sqrt{1 - (\frac{1}{2})^2}}{\sqrt{2\cdot\frac{1}{2} - 1}}$, Domain $= [\frac{1}{2}, 1] - \{\frac{1}{2}\}$ $= \underline{(\frac{1}{2}, 1]}$

$g(x) = \sqrt{2x - 1} = 0 \Rightarrow \boxed{x = \frac{1}{2}}$: we should exclude it!



Polynomial function : $f(x) = $(polynomial) Ex $f(x) = \underbrace{x^4 - 3x^2 + 4x - 1}_{\text{polynomial}}$

Algebraic function : $f(x) = $(algebraic expression) Ex $f(x) = \frac{x+3}{\sqrt{x-1}}$

Composition of Function

Given two function $f(x)$ and $g(x)$, the composite function

$(f \circ f)(x) = f(f(x))$

$\boxed{f \circ g}$ is defined as follows : $(f \circ g)(x) = f(g(x))$

$(g \circ f)(x) = g(f(x))$

$(f \circ g)(x) = ?$

$f(x) = 4x - 1$

$f(2) = 4 \cdot 2 - 1$, $f(g(x)) = 4 \cdot g(x) - 1$.

$f(f(x)) = 4f(x) - 1$

Ex If $f(x) = 4x - 1$ and $g(x) = 2x^2 + 1$, then

1) $(f \circ g)(x) = f(g(x)) = 4 \cdot g(x) - 1 = 4 \cdot (2x^2 + 1) - 1 = 8x^2 + 4 - 1$
$= 8x^2 + 3$

2) $(g \circ f)(x) = g(f(x)) = 2 \cdot f(x)^2 + 1 = 2 \cdot (4x-1)^2 + 1 = 2 \cdot (16x^2 - 8x + 1) + 1$
use $(a-b)^2 = a^2 - 2ab + b^2$
$= 32x^2 - 16x + 2 + 1$
$= 32x^2 - 16x + 3$

3) $(f \circ f)(x) = f(f(x)) = 4 \cdot f(x) - 1 = 4 \cdot (4x-1) - 1$
$= 16x - 4 - 1$
$= 16x - 5$

D.I.Y.

4) $(g \circ g)(x) = g(g(x)) = 2\, g(x)^2 + 1 = 2(2x^2+1)^2 + 1 = 2(4x^4 + 4x^2 + 1) + 1 = 8x^4 + 8x^2 + 2 + 1$
use $(a+b)^2 = a^2 + 2ab + b^2$
$= 8x^4 + 8x^2 + 3$

$(g \circ f)(1)$

5) $g(\boxed{f(1)}) = g(3) = \boxed{19}$

$f(x) = 4x - 1 \Rightarrow f(1) = 4 \cdot 1 - 1 = 4 - 1 = 3$.

$g(x) = 2x^2 + 1$
$g(3) = 2 \cdot 3^2 + 1 = 2 \cdot 9 + 1 = 18 + 1 = 19$

What can we say about the domain of the composite function?

Domain of $\boxed{f \circ g}$ is

$D(g) - \{$the set of all x such that f is undefind at $g(x)\}$.

Domain of g.

Ex Let $f(x) = x^2 - 4$ and $g(x) = \sqrt{x}$.

$D(g) = [0, \infty) - \text{nothing} = [0, \infty)$.

Ex) 3 is in $D(g)$.

$f(g(3)) = (g(3))^2 - 4$

$= (\sqrt{3})^2 - 4 = 3 - 4 = -1$.

$f(g(x)) = (g(x))^2 - 4$

1) Domain of $f \circ g$?

Wrong proof: $(f \circ g)(x) = f(g(x)) = g(x)^2 - 4 = (\sqrt{x})^2 - 4 = x - 4 \Rightarrow$ ~~Domain = $(-\infty, \infty)$~~

Ex $(f \circ g)(-1) = f(g(-1)) = f(\sqrt{-1})$: -1 is not in the domain!

not a real number

Correct proof: $D(g) = [0, \infty)$.

We need to exclude any points x in $D(g)$ such that f is undefined at $g(x)$.

However, the domain of f is all real number: $(-\infty, \infty)$

so f is always defined $\Rightarrow$ there is nothing to exclude.

Hence, the domain of $f \circ g$ is $[0, \infty)$

2) Domain of $g \circ f$? Will do it on Sep. 29.