Section 3.6 Continued.

Last time $\boxed{f(x)=ax^2+c}$

$\boxed{\text{HW5 : due this Friday at 1pm}}$

Given a quadratic function $f(x)=ax^2+bx+c$, we want to rewrite in the form $f(x)=a(x-h)^2+k$.

Ex $f(x)=-2x^2+4x+1$

$$=(-2x^2+4x)+1$$

$$=-2(x^2-2x)+1$$

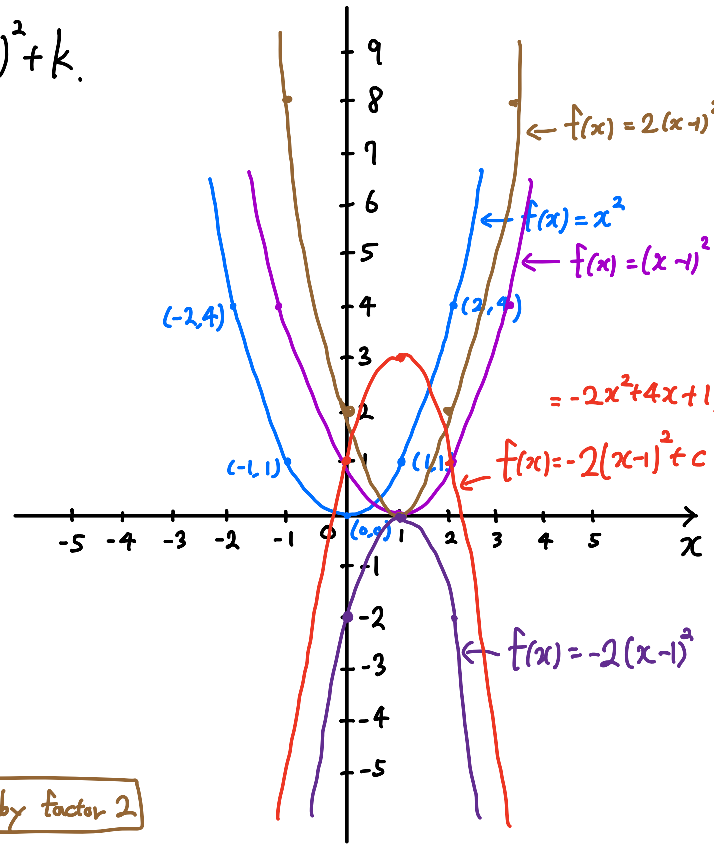
$$=2(\underbrace{x^2-2x+1}_{=(x-1)^2}-1)+1$$

$$=-2((x-1)^2-1)+1$$

$$=-2(x-1)^2+2+1$$

$$=-2(x-1)^2+3$$

$x^2-2x+b=(x+c)^2$
$=x^2+2cx+c^2$
$-2=2c,\ b=c^2$
$c=-1\quad b=(-1)^2=1$
$\Rightarrow x^2-2x+1=(x-1)^2$



Stretch the graph vertically by factor 2

$f(x)=x^2 \xrightarrow{\text{Move the graph} \rightarrow \text{1unit}} f(x)=(x-1)^2 \xrightarrow{\text{Stretch}} f(x)=2(x-1)^2 \xrightarrow{\text{Reflect the graph through x-axis}} f(x)=-2(x-1)^2 \xrightarrow{\text{move the graph} \uparrow \text{3units}} f(x)=-2(x-1)^2+3$

Move the graph → 1unit

Reflect the graph through x-axis

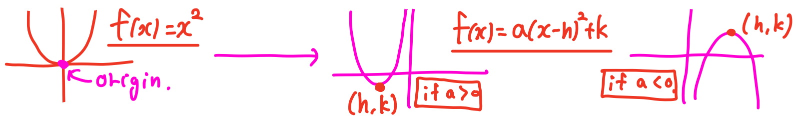
move the graph ↑ 3units

Standard Equation of a Parabola with Vertical Axis:

The graph of the equation $\boxed{y=a(x-h)^2+k}$ for $a\neq 0$ is a parabola that has $\boxed{\text{vertex } V(h,k)}$ and a vertical axis.

The parabola open upward if $a>0$ or downward if $a<0$.

Illustration :



Finding a standard equation of a parabola

Ex Express $y=-2x^2+8x+10$ as a standard equation of a parabola with a vertical axis. Find the vertex and sketch the graph.

$y=\boxed{-2}x^2+8x+10$

→ x-intercept : Set $y=0$: $0=-2x^2+8x+10$

$\downarrow \div(-2)$

$x^2-4x-5=0$

$(x-5)(x+1)=0$

$\downarrow$ Z.F.T.

$x-5=0$ or $x+1=0$

$x=5$ or $x=-1$

⇒ $(5,0)$ and $(-1,0)$

→ y-intercept : Set $x=0$: $y=10$.

⇒ $(0,10)$

$y=(-2x^2+8x)+10$

$y=-2(x^2-4x)+10$

$y=-2(x^2-4x+\overset{4}{b}-\overset{4}{b})+10$

$x^2-4x+b=(x+c)^2$

$=x^2+2cx+c^2$

$-4=2c \to c=-2$

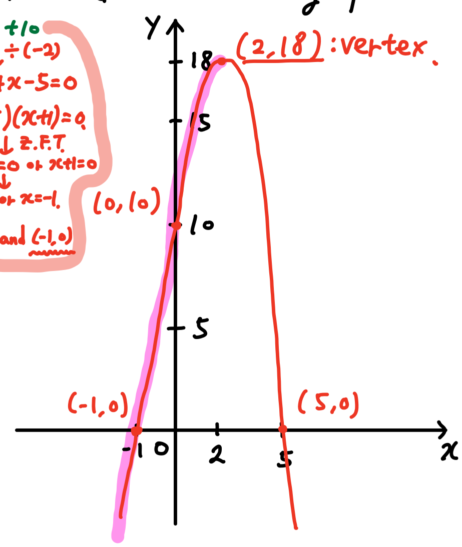
$b=c^2 \to b=4$

$x^2-4x+4=(x-2)^2$

$y=-2((x-2)^2-4)+10$

$y=-2(x-2)^2+8+10$

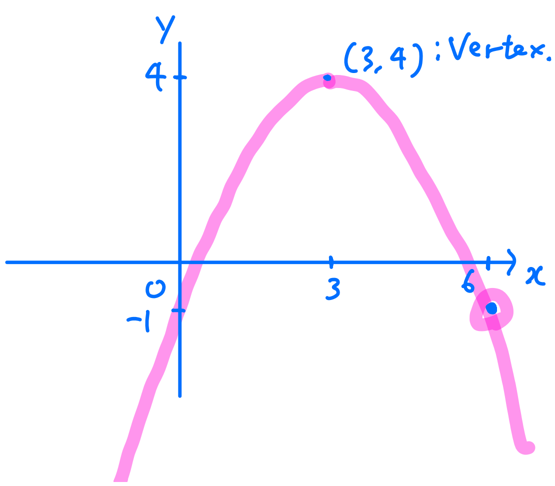
$y=-2(x-2)^2+18$. → Vertex : $(2,18)$



Domain : $(-\infty,\infty)$, Increasing : $(-\infty,2]$

Range : $(-\infty,18]$, Decreasing : $[2,\infty)$

Ex Find an equation of parabola that has vertex $V(3,4)$ and a vertical axis and passes through the point $(6,-1)$.



$y=a(x-\overset{3}{h})^2+\overset{4}{k}$.

: Vertex is (h,k). $h=3$, $k=4$

$\boxed{y=a(x-3)^2+4}$. $\cdots(*)$

$(6,-1)$ satisfies the equation $(*)$.

⇒ Replace x by 6, y by -1 in $(*)$

→ Replace a by $-\frac{5}{9}$ $\Rightarrow$ $\boxed{y=-\frac{5}{9}(x-3)^2+4}$

$-1=a\cdot(6-3)^2+4$
$-1=a\cdot 9+4$ → $9a=-5$
$a=\boxed{-\frac{5}{9}}$

Q : Given a parabola $y=ax^2+bx+c$ is there an easy way to find a vertex of it?

A : Yes!

Theorem for locating the Vertex of a Parabola:
The vertex of the parabola $\boxed{y=ax^2+bx+c}$ has x-coordinate $\boxed{-\frac{b}{2a}}$

Why? Let $y=ax^2+bx+c=(ax^2+bx)+c$

$$=a(x^2+\tfrac{b}{a}x)+c$$

$$=a\Big(\underline{x^2+\tfrac{b}{a}x+\tfrac{b^2}{4a^2}}-\tfrac{b^2}{4a^2}\Big)+c$$

$$=a\Big(\big(x+\tfrac{b}{2a}\big)^2-\tfrac{b^2}{4a^2}\Big)+c=a\big(x+\tfrac{b}{2a}\big)^2-\tfrac{b^2}{4a}+c$$

$\big(x-(-\frac{b}{2a})\big)^2$

x-coordinate of the vertex!

$x^2+\frac{b}{a}x+C=(x+d)^2$
$x^2+\frac{b}{a}x+C=x^2+2dx+d^2$
$2d=\frac{b}{a}$ $\Rightarrow$ $d=\frac{b}{2a}$
$C=d^2$ $\quad C=(\frac{b}{2a})^2=\frac{b^2}{4a^2}$
$\Rightarrow x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=(x+\frac{b}{2a})^2$

<u>Ex</u> Find the vertex of the parabola $y=\underset{a}{3}x^2-\underset{b}{4}x+\underset{c}{2}$

By the theorem $-\frac{b}{2a}=-\frac{(-4)}{2\cdot 3}=\frac{4}{6}=\frac{2}{3}$ is the x-coordinate of the vertex.

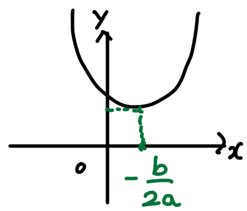
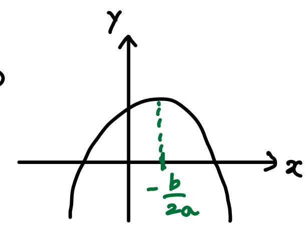
Replace x by $\frac{2}{3}$ $\Rightarrow$ $y=3\cdot(\frac{2}{3})^2-4\cdot(\frac{2}{3})+2$

$$=\cancel{3}\cdot\frac{4}{\cancel{9}_3}-\frac{8}{3}+2=\frac{4}{3}-\frac{8}{3}+\frac{6}{3}=\frac{2}{3}$$

$\Rightarrow$ Vertex is $\boxed{\left(\frac{2}{3},\frac{2}{3}\right)}$

What can we say about the vertex of the quadratic function

$f(x) = ax^2+bx+c$?

If $a>0$  , if $a<0$ 

From the previous theorem, we know that the x-coordinate of the vertex of $y = ax^2+bx+c$ is $-\frac{b}{2a}$.

Hence, we have ...

Theorem on the Maximum or Minimum Value of a Quadratic Function:

If $f(x) = ax^2+bx+c$, where $a \neq 0$, then $f(-\frac{b}{2a})$ is

(1) the maximum value of f if $\underline{a<0}$

(2) the minimum value of f if $a>0$

__Ex__ Find the maximum value of the function $f(x) = -x^2+5x+7$

$a=-1$ $b=5$ $c=7$.

$a<0$

$$x = -\frac{b}{2a} = -\frac{5}{2\cdot(-1)} = -\frac{5}{-2} = \frac{5}{2}$$

$$f\left(\frac{5}{2}\right) = -\left(\frac{5}{2}\right)^2 + 5\cdot\frac{5}{2} + 7 = -\frac{25}{4} + \frac{25}{2} + 7 = -\frac{25}{4} + \frac{50}{4} + \frac{28}{4} = \boxed{\frac{53}{4}}$$

Maximum because $a=-1<0$.