Section 2.3 Continued

* (Written / On line) Homework 2 : due this Friday at 1 pm.

Last time : $(x+m)^2 = n \rightarrow x+m = \pm\sqrt{n} \rightarrow x = -m \pm \sqrt{n}$

$(x+m)^2 = n$

General quadratic equation can be written in this form.

To do that, we need to "complete the square".

: $x^2 + kx \longrightarrow x^2 + kx + \left(\frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2$

$\Rightarrow \boxed{\left(x + \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2}$

$x^2 - kx \longrightarrow x^2 - kx + \left(\frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2$

$\Rightarrow \boxed{\left(x - \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2}$

$\left((x+c)^2 = x^2 + 2cx + c^2 \right.$

$\left(x + \frac{k}{2}\right)^2 = x^2 + kx + \left(\frac{k}{2}\right)^2$

$\left. 2c = k,\ c = \frac{k}{2} \right)$

Determine the value or values of d that complete the square for each expression.

Write the trinomial and the square of the binomial it represents.

<u>Ex</u> $x^2 + 5x + d = (x+c)^2$

$= x^2 + 2cx + \cancel{c^2}\ \frac{25}{4}$

① Compare the coefficients of x

$2c = 5,\ c = \frac{5}{2}$

② Compare the constant terms.

$d = c^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

$\Rightarrow d = \boxed{\frac{25}{4}}$,

: $\boxed{x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2}$

<u>Ex</u> $x^2 + dx + 25 = (x+c)^2$

$= x^2 + 2cx + c^2$

① Compare the constant terms

$c^2 = 25.$

$c = +5$ or -5

② Compare the coefficients of x.

$d = 2c. \Rightarrow d = +10$ or -10.

when, $\boxed{d = +10 : x^2 + 10x + 25 = (x+5)^2}$

$\boxed{d = -10 : x^2 - 10x + 25 = (x-5)^2}$.

Using "Completing the square", we can solve many (in fact, all) quadratic equations.

Ex Solve : $x^2 + 3x - 7 = 0$

$$x^2 + 3x - 7 = 0 \quad +7 \quad +7$$

$$x^2 + 3x = 7$$

$$+\left(\tfrac{3}{2}\right)^2 \quad +\left(\tfrac{3}{2}\right)^2$$

$$x^2 + 3x + \left(\tfrac{3}{2}\right)^2 = 7 + \left(\tfrac{3}{2}\right)^2$$

$$x^2 + 2\cdot x \cdot \tfrac{3}{2} + \left(\tfrac{3}{2}\right)^2 = 7 + \tfrac{9}{4}$$

* Recall : $x^2 + 2\cdot x \cdot y + y^2 = (x+y)^2$

$$\left(x + \tfrac{3}{2}\right)^2 = \tfrac{28}{4} + \tfrac{9}{4}$$

$$\left(x + \tfrac{3}{2}\right)^2 = \frac{37}{4}$$

$$\left(x + \tfrac{3}{2}\right) = \pm\sqrt{\frac{37}{4}} = \pm\frac{\sqrt{37}}{\sqrt{4}} = \pm\frac{\sqrt{37}}{2}$$

$$-\tfrac{3}{2} \qquad -\tfrac{3}{2}$$

$$\Rightarrow \boxed{x = -\frac{3}{2} \pm \frac{\sqrt{37}}{2}}$$

Using the proof of the above example, we can prove "Quadraic Formula":

If $a \neq 0$, the roots of $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(Proof) $ax^2 + bx + c = 0$

$$\downarrow \div a$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

↓ complete the square

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ discriminant! (usually denoted by D)

The quadratic equation $ax^2+bx+c=0$ has

① Two real and unequal roots if $D>0$.

② One root of multiplicity 2 if $D=0$.

③ No real root if $D<0$ (It has two complex and unequal roots).

Ex 1. $x^2-4x+3=0 \Rightarrow x = \frac{-(-4)\pm\sqrt{(-4)^2-4\cdot1\cdot3}}{2\cdot1} = \frac{4\pm\sqrt{4}}{2} = \frac{4\pm2}{2}$ $\left\langle \frac{4+2}{2}=\frac{6}{2}=3,\ \frac{4-2}{2}=\frac{2}{2}=1 \right.$

$x^2-4x+4=0 \Rightarrow x = \frac{-(-4)\pm\sqrt{(-4)^2-4\cdot1\cdot4}}{2\cdot1} = \frac{4\pm\sqrt{0}}{2} = \frac{4\pm0}{2}$ $\left\langle \frac{4+0}{2}=2,\ \frac{4-0}{2}=2 \right.$

$x^2-4x+5=0 \Rightarrow x = \frac{-(-4)\pm\sqrt{(-4)^2-4\cdot1\cdot5}}{2\cdot1} = \frac{4\pm\sqrt{-4}}{2} \Rightarrow$ No real solutions

NOT a real number

Ex Solve $10x = 3x^2-8$ using the quadratic equation.

$-10x \qquad -10x$

$0 = 3x^2-8-10x$

$0 = \underset{a}{3}x^2 \underset{b}{-10}x \underset{c}{-8}$

$\to x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow x = \frac{-(-10)\pm\sqrt{(-10)^2-4\cdot3\cdot(-8)}}{2\cdot3}$

$= \frac{10\pm\sqrt{100+96}}{6}$

$= \frac{10\pm\sqrt{196}}{6}$

$= \frac{10\pm14}{6}$

$= \frac{10+14}{6}$ or $\frac{10-14}{6}$

$= \frac{24}{6}$ or $\frac{-4}{6}$

$= \boxed{4 \text{ or } -\frac{2}{3}}$

Ex Factor: $\boxed{3x^2-10x-8}$

* Let $f(x)=ax^2+bx+c$ be a quadratic polynomial and p, q are the solutions of $f(x)=0$. Then $f(x)=a(x-p)(x-q)$

$\Rightarrow 3x^2-10x-8 = 3(x-4)(x-(-\frac{2}{3})) = 3(x-4)(x+\frac{2}{3}) = (x-4)\cdot3(x+\frac{2}{3})$

$= \boxed{(x-4)(3x+2)}$

Ex Factor $6x^2-7x-20$ (Recall: $529=23^2$)

Proof 1) If you can do it using trial & error method, it is great: $\boxed{(3x+4)(2x-5)}$

Proof 2) If not, you can still solve it by solving $6x^2-7x-20=0$.

$$6x^2-7x-20=0$$

$$\Rightarrow x=\frac{-(-7)\pm\sqrt{(-7)^2-4\cdot6\cdot(-20)}}{2\cdot6}=\frac{7\pm\sqrt{49+480}}{12}=\frac{7\pm\sqrt{529}}{12}=\frac{7\pm23}{12}=\frac{7+23}{12}\text{ or }\frac{7-23}{12}$$

$\frac{30}{12}=\frac{5}{2}$ $\qquad$ $\frac{-16}{12}=-\frac{4}{3}$

Hence, $6x^2-7x-20=6\left(x-\frac{5}{2}\right)\left(x-\left(-\frac{4}{3}\right)\right)=6\left(x-\frac{5}{2}\right)\left(x+\frac{4}{3}\right)=3\cdot2\cdot\left(x+\frac{4}{3}\right)\left(x-\frac{5}{2}\right)$

$=3\cdot\left(x+\frac{4}{3}\right)\cdot2\cdot\left(x-\frac{5}{2}\right)$

$=\boxed{(3x+4)(2x-5)}$

Solving rational equation

Ex $\frac{3x}{x+4}-\frac{x+1}{x-2}=\frac{2x-22}{x^2+2x-8}$

$\Rightarrow \frac{3x}{x+4}-\frac{x+1}{x-2}=\frac{2x-22}{(x+4)(x-2)}.$

$\downarrow$ multiply $(x+4)(x-2)$

$\cancel{(x+4)}(x-2)\frac{3x}{\cancel{x+4}}-(x+4)\cancel{(x-2)}\frac{x+1}{\cancel{x-2}}=\cancel{(x+4)}\cancel{(x-2)}\frac{2x-22}{\cancel{(x+4)}\cancel{(x-2)}}.$

$(x-2)\cdot3x-(x+4)(x+1)=2x-22$

$3x^2-6x-(x^2+5x+4)=2x-22$

$3x^2-6x-x^2-5x-4=2x-22$

$2x^2-11x-4=2x-22$

$\qquad -(2x-22) \qquad -(2x-22)$

$2x^2-11x-4-2x+22=0$

$2x^2-13x+18=0.$

$(2x-9)(x-2)=0.$

$\downarrow$ Z.F.T.

$2x-9=0$ or $x-2=0.$

$\boxed{x=\frac{9}{2}}$ or $\cancel{x=2}$

$x=2$ is an extraneous solution because two denominators ($x-2$ and x^2+2x-8) become zero when $x=2$: NOT a solution!

☆ Recall : We treat all other variables as if they are constants! ☆

Solving an equation with more than one variable for one variable.

Ex Solve $y = x^2 + 4x + 3$ for x, where $x \geq -2$

subtract y

$0 = \underbrace{1}_{a} \cdot x^2 \underbrace{+ 4}_{b} x \underbrace{+ (3-y)}_{c}$

By quadratic formula,

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 (3-y)}}{2 \cdot 1}$$

$$= \frac{-4 \pm \sqrt{16 - 4(3-y)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 12 + 4y}}{2}$$

$$= \frac{-4 \pm \sqrt{4 + 4y}}{2}$$

$$\frac{-4 \pm \sqrt{4(1+y)}}{2}$$

$$= \frac{-4 \pm 2\sqrt{1+y}}{2}$$

$$= -\frac{4}{2} \pm \frac{2\sqrt{1+y}}{2}$$

$$= -2 \pm \sqrt{1+y}$$

$$= \boxed{-2 + \sqrt{1+y}} \text{ or } \cancel{-2 - \sqrt{1+y}}$$

This is because $x \geq -2$

Had no time to show this example in the class. Please do it your own and compare it with my answer!

Ex Solve $2x^2 - 3y^2 = xy + 2$ for (1) x
(2) y.

* The proof is on the next page!

(Please try the problem on your own first!)

(1) Solve $2x^2 - 3y^2 = xy + 2$ for x.

$$2x^2 - 3y^2 - (xy+2) = \cancel{xy+2} - \cancel{(xy+2)}$$

$$2x^2 - 3y^2 - xy - 2 = 0$$

$$\underbrace{2}_{a}x^2 \underbrace{- y}_{b}\cdot x \underbrace{- 3y^2 - 2}_{c} = 0$$: A quadratic equation in x!

By quadratic formula: $$x = \frac{-(-y) \pm \sqrt{(-y)^2 - 4\cdot 2\cdot(-3y^2-2)}}{2\cdot 2} = \frac{y \pm \sqrt{y^2 - 8(-3y^2-2)}}{4}$$

$$= \frac{y \pm \sqrt{y^2 + 24y^2 + 16}}{4}$$

$$= \boxed{\frac{y \pm \sqrt{25y^2+16}}{4}}$$

(2) Solve $2x^2 - 3y^2 = xy + 2$ for y.

$$\cancel{(2x^2-3y^2)} - \cancel{(2x^2-3y^2)} = xy + 2 - (2x^2 - 3y^2)$$

$$0 = xy + 2 - 2x^2 + 3y^2$$

$$0 = \underbrace{3}_{a}y^2 + \underbrace{x}_{b}\cdot y + \underbrace{2 - 2x^2}_{c}$$: A quadratic equation in y!

By quadratic formula: $$y = \frac{-x \pm \sqrt{x^2 - 4\cdot 3(2-2x^2)}}{2\cdot 3} = \frac{-x \pm \sqrt{x^2 - 12(2-2x^2)}}{6}$$

$$= \frac{-x \pm \sqrt{x^2 - 24 + 24x^2}}{6}$$

$$= \boxed{\frac{-x \pm \sqrt{25x^2 - 24}}{6}}$$