Section 1.4. Continued

Rationalizing a denominator / a numerator. → removing radical symbols!

(* use $(x+y)(x-y) = x^2 - y^2$ or $(x+y)(x^2-xy+y^2) = x^3+y^3$
$(x-y)(x^2+xy+y^2) = x^3-y^3$)

$\sqrt[3]{x}$ $\sqrt[3]{a}$ $\sqrt[3]{x}$ $\sqrt[3]{x}\sqrt[3]{a}$ $\sqrt[3]{a}$ $\sqrt[3]{x}$ $\sqrt[3]{a}$ → $(\sqrt[3]{x}-\sqrt[3]{a})\left((\sqrt[3]{x})^2+\sqrt[3]{x}\cdot\sqrt[3]{a}+(\sqrt[3]{a})^2\right)$
$= (\sqrt[3]{x})^3 - (\sqrt[3]{a})^3 = x - a$

Ex $\dfrac{1}{\sqrt{x}-\sqrt{y}} \cdot \dfrac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})} = \dfrac{\sqrt{x}+\sqrt{y}}{(\sqrt{x})^2-(\sqrt{y})^2}$
$= \boxed{\dfrac{\sqrt{x}+\sqrt{y}}{x-y}}$

Ex Rationalize the denominator : $\dfrac{1}{\sqrt[3]{x}-\sqrt[3]{a}} \times \dfrac{\left((\sqrt[3]{x})^2+\sqrt[3]{x}\cdot\sqrt[3]{a}+(\sqrt[3]{a})^2\right)}{\left((\sqrt[3]{x})^2+\sqrt[3]{x}\cdot\sqrt[3]{a}+(\sqrt[3]{a})^2\right)}$

$= \boxed{\dfrac{(\sqrt[3]{x})^2+\sqrt[3]{x}\cdot\sqrt[3]{a}+(\sqrt[3]{a})^2}{x-a}} = \boxed{\dfrac{\sqrt[3]{x^2}+\sqrt[3]{xa}+\sqrt[3]{a^2}}{x-a}}$

Ex If $h \neq 0$, rationalize the numerator : $\dfrac{\sqrt{2x+h}-\sqrt{2x}}{h} \times \dfrac{(\sqrt{2x+h}+\sqrt{2x})}{(\sqrt{2x+h}+\sqrt{2x})}$

$(\sqrt{2x+h}-\sqrt{2x}) \times (\sqrt{2x+h}+\sqrt{2x})$
$= (\sqrt{2x+h})^2 - (\sqrt{2x})^2 = 2x+h-2x = h$

$= \dfrac{\cancel{h}^{1}}{1\,\cancel{h}(\sqrt{2x+h}+\sqrt{2x})} = \boxed{\dfrac{1}{\sqrt{2x+h}+\sqrt{2x}}}$

Ex Express the following as a quotient : $x^{-2}+x^{-5}$

$\boxed{\text{Recall: } a^n = \dfrac{1}{a^{-n}},\ a^{-n} = \dfrac{1}{a^n}}$

$\dfrac{x^2 : x^2}{x^5 : x^5}$ l.c.d. $= x^5$

$= \dfrac{1\cdot x^3}{x^2 x^3} + \dfrac{1}{x^5} = \dfrac{x^3}{x^5} + \dfrac{1}{x^5} = \boxed{\dfrac{x^3+1}{x^5}}$

Ex Simplify $\dfrac{\boxed{(2)\cdot(x^3-2)^{\frac{1}{3}} - (2x)\left(\frac{1}{3}\right)(x^3-2)^{-\frac{2}{3}}(3x^2)}}{\left[(x^3-2)^{\frac{1}{3}}\right]^2} = \dfrac{\frac{-4}{(x^3-2)^{\frac{2}{3}}}}{\left[(x^3-2)^{\frac{1}{3}}\right]^2}$

$\boxed{\text{derivative of } \dfrac{2x}{(x^3-2)^{\frac{1}{3}}} = \dfrac{d}{dx}\left(\dfrac{2x}{(x^3-2)^{\frac{1}{3}}}\right)}$

$= \dfrac{\frac{-4}{(x^3-2)^{\frac{2}{3}}}}{\frac{(x^3-2)^{\frac{2}{3}}}{1}} = \boxed{\dfrac{-4}{(x^3-2)^{\frac{4}{3}}}}$

First, simplify the top : $(2)(x^3-2)^{\frac{1}{3}} - (2x)\cdot\left(\frac{1}{\cancel{3}}\right)\cdot(x^3-2)^{-\frac{2}{3}}\cdot(\cancel{3}x^2)$

$= (2)(x^3-2)^{\frac{1}{3}} - 2x^3\,(x^3-2)^{-\frac{2}{3}}$

$= (2)(x^3-2)^{\frac{1}{3}} - 2x^3\,\dfrac{1}{(x^3-2)^{\frac{2}{3}}}$

$= \dfrac{(2)(x^3-2)^{\frac{1}{3}} \times (x^3-2)^{\frac{2}{3}}}{1 \times (x^3-2)^{\frac{2}{3}}} - \dfrac{2x^3}{(x^3-2)^{\frac{2}{3}}}$

$= \dfrac{2(x^3-2)}{(x^3-2)^{\frac{2}{3}}} - \dfrac{2x^3}{(x^3-2)^{\frac{2}{3}}}$

$= \dfrac{2(x^3-2)-2x^3}{(x^3-2)^{\frac{2}{3}}} = \dfrac{\cancel{2x^3}-4-\cancel{2x^3}}{(x^3-2)^{\frac{2}{3}}} = \boxed{\dfrac{-4}{(x^3-2)^{\frac{2}{3}}}}$

This is a numerator!

recall : $(a^m)^n = a^{mn}$
recall : $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{ad}{bc}$

- Do the following Exercises by your own!

(Proof will be provided when I upload the lecture note)

③ Simplify $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

We simplify the numerator first.
To do that rewrite $\frac{1}{(x+h)^2}$ and $\frac{1}{x^2}$ using l.c.d. $= x^2(x+h)^2$.

$$= \frac{\dfrac{1\cdot x^2}{(x+h)^2\cdot x^2}-\dfrac{1\cdot(x+h)^2}{x^2\cdot(x+h)^2}}{h} = \frac{\dfrac{x^2}{x^2(x+h)^2}-\dfrac{(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \frac{\dfrac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} = \frac{\dfrac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}}{h} = \frac{\dfrac{-2xh-h^2}{x^2(x+h)^2}}{h} = \frac{\dfrac{-h(2x+h)}{x^2(x+h)^2}}{\dfrac{h}{1}}$$

I used the fact that $h=\frac{h}{1}$

Here, I used $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{ad}{bc}$

$$= \frac{-\cancel{h}(2x+h)}{\cancel{h}\,x^2(x+h)^2} = \boxed{-\frac{2x+h}{x^2(x+h)^2}}$$

④ Rationalize the denominator: $\dfrac{\sqrt{x}-3}{\sqrt{x}+3} = \dfrac{(\sqrt{x}-3)(\sqrt{x}-3)}{(\sqrt{x}+3)(\sqrt{x}-3)} = \dfrac{(\sqrt{x}-3)^2}{(\sqrt{x})^2-3^2} = \boxed{\dfrac{x-6\sqrt{x}+9}{x-9}}$

denominator is the sum of $\sqrt{x}$ and 3,
so I multiply the difference $\sqrt{x}-3$ both on the top and the bottom!

⑤ Simplify $\dfrac{2\cdot(x^2-3)^{\frac{1}{2}}-(2x)\cdot\left(\frac{1}{2}\right)\cdot(x^2-3)^{-\frac{1}{2}}\cdot(2x)}{\left[(x^2-3)^{\frac{1}{2}}\right]^2} = \dfrac{\dfrac{-6}{(x^2-3)^{\frac{1}{2}}}}{(x^2-3)} = \dfrac{\dfrac{-6}{(x^2-3)^{\frac{1}{2}}}}{\dfrac{(x^2-3)}{1}}$

$$= \frac{-6}{(x^2-3)^{\frac{1}{2}+1}}$$

$$= \boxed{\frac{-6}{(x^2-3)^{\frac{3}{2}}}}$$

Similar to the last example in the previous page we first simplify the numerator:

$2\cdot(x^2-3)^{\frac{1}{2}}-(\cancel{2}x)\cdot(\frac{1}{\cancel{2}})\cdot(x^2-3)^{-\frac{1}{2}}\cdot(2x)$

$= 2\cdot(x^2-3)^{\frac{1}{2}}-2x^2\cdot(x^2-3)^{-\frac{1}{2}}$

$= 2\cdot(x^2-3)^{\frac{1}{2}}-2x^2\cdot\dfrac{1}{(x^2-3)^{\frac{1}{2}}}$

$= \dfrac{2\cdot(x^2-3)^{\frac{1}{2}}}{1}-\dfrac{2x^2}{(x^2-3)^{\frac{1}{2}}}$

$= \dfrac{2\cdot(x^2-3)^{\frac{1}{2}}(x^2-3)^{\frac{1}{2}}}{1\cdot(x^2-3)^{\frac{1}{2}}}-\dfrac{2x^2}{(x^2-3)^{\frac{1}{2}}}$

$= \dfrac{2\cdot(x^2-3)}{(x^2-3)^{\frac{1}{2}}}-\dfrac{2x^2}{(x^2-3)^{\frac{1}{2}}}$

$= \dfrac{2\cdot(x^2-3)-2x^2}{(x^2-3)^{\frac{1}{2}}}$

$= \dfrac{2x^2-6-2x^2}{(x^2-3)^{\frac{1}{2}}}$

$= \boxed{\dfrac{-6}{(x^2-3)^{\frac{1}{2}}}}$: This is the numerator of the original expression after the simplification!

Remark) The given expression is a derivative of $\dfrac{2x}{(x^2-3)^{\frac{1}{2}}}$, denoted by $\dfrac{d}{dx}\left(\dfrac{2x}{(x^2-3)^{\frac{1}{2}}}\right)$.
You will learn about the derivative in M211!

Chapter 2. Equations.

"Equation in one variable x"

is a statement of equality involving one variable x.

Ex $2x=8$; $x=4$ is the solution.

"Solution" of an equation in x

is a number that yields a true statement when we replace

x by the number.

"Solving an equation" is a journey to find the solutions

of the given equation.

To solve the equation (or find the solution), we are allowed to use the following properties of

If c is any real number, then

$$a=b \longleftrightarrow a+c=b+c$$

$$a=b \longleftrightarrow a-c=b-c$$

If c is any non-zero real number, then

$$a=b \longleftrightarrow a\cdot c=b\cdot c$$

$$a=b \longleftrightarrow \frac{a}{c}=\frac{b}{c}$$

$2x=8$

$\downarrow$

$2x\cdot 0=8\cdot 0$

$\boxed{0=0}$

: Never multiply 0!

If an equation in x is true for every x, we call the equation an identity.

ex) $2x+1=2x+1$: every x satisfy this equation, so it is an identity.

Otherwise, it is called conditional equation.

In this section, all the equations in x are equivalent to a certain linear equation

$ax+b=0$.

Ex Solve: $5x - 11 = 2x + 7$

$$5x - 11 - 2x = \cancel{2x} + 7 - \cancel{2x}$$

$$3x - 11 = 7$$

$$3x \cancel{-11} \cancel{+11} = 7 + 11$$

$$\frac{3x}{3} = \frac{18}{3}$$

$$\boxed{x = 6}$$

(Check) $5 \cdot 6 - 11 = 2 \cdot 6 + 7$

$30 - 11 = 12 + 7,\ \boxed{19 = 19}$ ✓

Ex Solve: $(4x - 3)(5x + 2) = (10x + 5)(2x + 1)$

FOIL

$$20x^2 + 8x - 15x - 6 = 20x^2 + 10x + 10x + 5$$

$$20x^2 - 7x - 6 = 20x^2 + 20x + 5$$

$-20x^2$

$$-7x - 6 = 20x + 5$$

$+7x$

$$-6 = 27x + 5$$

-5

$$-11 = 27x \xrightarrow{\div 27} \boxed{x = -\frac{11}{27}}$$

(Check) We did not check it in the class, so let me do it now.

$$\left(4 \cdot \left(-\frac{11}{27}\right) - 3\right)\left(5 \cdot \left(-\frac{11}{27}\right) + 2\right) = \left(10 \cdot \left(-\frac{11}{27}\right) + 5\right)\left(2 \cdot \left(-\frac{11}{27}\right) + 1\right)$$

$$\left(-\frac{44}{27} - 3\right)\left(-\frac{55}{27} + 2\right) = \left(-\frac{110}{27} + 5\right)\left(-\frac{22}{27} + 1\right)$$

$$\left(-\frac{44}{27} - \frac{81}{27}\right)\left(-\frac{55}{27} + \frac{54}{27}\right) = \left(-\frac{110}{27} + \frac{135}{27}\right)\left(-\frac{22}{27} + \frac{27}{27}\right)$$

$$\left(-\frac{125}{27}\right)\left(-\frac{1}{27}\right) = \frac{25}{27} \cdot \frac{5}{27}$$

$$\boxed{\frac{125}{729} = \frac{125}{729}}$$ ✓